Washington University

John M. Olin School of Business

Dissertation Examination Committee

Panos Kouvelis (Chair);
Guofu Zhou
Yossi Aviv
Tava Lennon Olsen
Joseph Milner
Lingxiu Dong

Essays in Global Supply Chain and Revenue Management in the Network Economy

By

Qing Ding

A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

August 2004

St. Louis, Missouri

Acknowledgements

I take this opportunity to thank all of those individuals who helped me prepare this dissertation. First, I want to express my deep appreciation to my advisor, Panos Kouvelis. His continuous support and encouragement throughout the period of my doctoral program have played an irreplaceable role in my growth as an independent and mature researcher. I am also grateful to Joseph Milner and Lingxiu Dong for their insightful comments and suggestions on my research during various stages. In addition, I would particularly like to thank Guofu Zhou, Yossi Aviv and Tava Lennon Olsen for providing very helpful feedback and serving on my committee.

Special thanks Judy Van Dyke, Marie Lay and Molly B. Shaikewitz for their administrative support. I would like to thank Julie Niederhoff, Ping Su and Zhenxin (Dennis) Yu for many useful discussions, and Jun Tu, Yu Ma, Yuanfang Lin and Fenghua Song for their constant encouragement and for sharing my happiness and stress.

Finally, a big thank you from the bottom of my heart goes to my wife, Qihong Xie, and my daughter, Mengyao Ding, for accompanying me through the most difficult times with love and great support.

Dissertation Abstract

My dissertation research includes the essays on two topics: (a). Integrated risk management (combined Operational and Financial Hedging) approaches to global production planning issues (co-work with Lingxiu Dong and Panos Kouvelis); (b). Dynamic revenue management approaches to optimizing multi-class customer demand fulfillment (co-work with Panos Kouvelis and Joseph Milner).

In the risk management essays, we study the implications of financial hedging policies on location and production planning decisions of risk averse global firms in the presence of demand and exchange rate uncertainty. We consider that a firm can choose production location between his home country and a foreign country and sell to both local and foreign markets. In the first stage a capacity, a financial hedging contract and the location of production center are decided in the presence of demand and exchange rate uncertainty. In the second stage, after the realization of demand and exchange rate, a production "allocation" decision (e.g., how many units to localize in home market and distribute to the foreign market) is made in order to optimize profits. The second stage allocation decision (referred to as an "allocation" option) is the firm's real option serving as an operational hedge of the demand and exchange rate uncertainty. We want to understand the role of the allocation option and financial hedging on the production location and planning decisions.

For a stochastic foreign market, given a capacity and using a group of call option currency contracts at the prespecified exercise prices, the optimal

hedging sizes are the unique solutions to the linear equations. The optimal capacity is uniquely determined by close formulation for the independent case (i.e. demand being independent with exchange rate) when using a single call option with exercise price equal to the ratio of the localization cost and selling price (i.e. τ/p).

Our analysis clearly shows that the financial hedge consists of two separated parts: one minimizing the profit variance and the other balancing the cost of purchasing financial hedging contracts (e.g. the "risk premium"). The risk premium is small in the most practical environments and its effect on hedging size is independent with production decisions. Thus, we present strong results by ignoring the risk premium and focusing on minimizing the profit variance. For the independent case with zero risk premium, the optimal financial hedging policy, among all combinations of call options, put options and forwards contracts, is a single call option with exercise price equal to τ/p. The optimal hedging size and capacity are uniquely determined by the close formulations.

When demand and exchange rates are correlated, the optimal hedging is complex. On the other hand, for most of the cases, well designed call option currency contracts, as prescribed by our formulas, prove very effective in controlling total risk.

Our analysis clearly establishes the values of allocation option and financial hedging for risk averse firms. For the independent case with no financial hedging, the use of an allocation option favors increased production, in an effort to improve expected profits and the objective value. Similarly, the use of a financial hedge, as intuitively expected, allows an increased capacity to improve expected profits and the objective value, through better control of profit variance, no matter using or no using an allocation option, if the risk premium is positive. For the cases using financial hedge, using allocation option increases capacity, and improves expected profit and the objective value if ignoring the risk premium.

For a stochastic domestic market and a stochastic foreign market, the optimal hedge size vector is the unique, given a capacity and using a group of call option currency contracts at the prespecified exercise prices.

For the independent case with zero risk premium, among any combination of call options, put options and forward contracts, the optimal hedge consists of the two call option currency contracts, at one exercise price equal to τ/p and another equal to the per unit profit at the local market. However, there may exist multiple local optimal capacities. The financial hedging increases the capacity for the independent case in the presence of allocation option, and increases the capacity for the foreign market when no using allocation option, but the allocation option may decrease capacity even if the demands are known constants.

Furthermore, both the allocation option and the financial hedge may shifts the location of the production center from domestic to foreign or from foreign to domestic even for the independent cases.

In the revenue management essays, we are concerned with the problem of allocating inventory over a horizon to demand from several classes of customers when partial backlogging of unfulfilled demand is possible. The customers are distinguished into several classes by the price they are to pay for the item. The probability of customers' commitment to wait is influenced by a discount the firm may offer as well as some class specific parameters.

In the first essay, demand from each customer class is modeled as a realization of a (non-stationary) random variable during each of several stages a period is divided into. The firm is able to view this demand in each stage prior to making an allocation decision on which demand to fill. Unfilled demand may then be delivered at the later stages or the beginning of the next period.

We present a solution approach to the problem of determining the inventory allocation, the customer discounts and the prioritization of demand for all stages (referred to as the ADP problem), through dynamic programming starting first with the final stage and then solving the problem by induction. We show that a class order policy is optimal with waiting customers being served subsequent to new demand in any stage. We describe how to find the inventory allocation in any stage for each class, with the use of class specific threshold limits. The class based threshold limits are monotonic in the waiting demand and the inventory remaining. We also discuss the optimal determination of dynamically offered discounts depending on the available inventory and realized demand for the stage. For the continuous time demand case, we provide an efficient and robust heuristic for the solution of the ADP problem in real time.

Our numerical results clearly indicate the advantageous shifting of the inventory-service frontier through the implementation of ADP policies. The ADP solution always increases the expected profit vis-à-vis the FCFS (First come / first serve) or no price discounting policies, often in the range of 15-20%. Through appropriate discounting that leads to significant customer retention and increases demand waiting rates of price sensitive customers, ADP policies reduce the base stock levels as well as the incurred holding costs in almost all cases while increasing the total fill rate (i.e. percentage of demand eventually satisfied) of all classes. ADP policies also improve the prompt fill rates (i.e. percentage of demand immediately satisfied) of the most profitable and time sensitive customers.

In the second revenue management essay, we focus on the deterministic demand problem with constant arriving rates on the infinite horizon problem (a generalized "EOQ" model).

We propose an optimal rationing and discounting policy for the EOQ model. After introducing the concept of prompt service welfare, it is intuitive to find the optimal price discounts and the optimal rationing policy given the cycle time. Then, the optimal order policy is determined directly from the first order condition on the cycle time. The numerical analysis illustrates that both inventory rationing and price discounting can increase the average profit and the customer fill rates significantly by comparing the results to those from the no discounting policy and the naïve FCFS policy.

Contents

Acknowledgements ii

Dissertation Abstract iii

1. On the Interaction of Production and Financial Hedging Decisions in Global Markets 1

1.1 Introduction 2

1.1.1 Problem Motivation 2

1.1.2 Literature Review 3

1.1.3 Paper Organization 6

1.2 Model 7

1.3 Hedging the Currency Risk of a Stochastic Demand Foreign Market 12

1.3.1 Problem Solution 12

1.3.2 Solution of the Problem with Zero Risk Premium 18

1.3.3 Role of Production and Financial Hedges 22

1.3.4 Numerical Analysis and Managerial Insights 26

1.4 Hedging the Currency Risk of a Stochastic Demand Domestic Market and a Stochastic Demand Foreign Market 32

1.4.1 Problem Solution 32

1.4.2 Solution of the Problem with Zero Risk Premium 34

1.4.3 Role of Production and Financial Hedges 39

1.4.3.1 Impact of Production Allocation 39

1.4.3.2 Impact of Financial Hedging 42

1.4.3.3 Comparison of Production Allocation and Financial Hedging........47

1.5 Foreign Production Center Problem 49
1.5.1 Model and Solution 49
1.5.2 Impact of Allocation Option 49
1.5.3 Impact of Financial Hedging 50
1.6 Managerial Insights & Conclusions 53
1.7 References.. 56

2. Dynamic Pricing through Customer Discounts for Optimizing Multi-Class Customers Demand Fulfillment 59
2.1 Introduction 60
2.1.1 Problem Motivation 60
2.1.2 Literature Review 61
2.1.3 Paper Organization 64
2.2 Model 65
2.3 ADP Problem Solution 69
2.3.1 Stage *M* 69
2.3.2 Stage *j* 73
2.3.3 Expected Demand Heuristic... 82
2.4 A Continuous Time Heuristic 86
2.5 A Comparison of the ADP Solution to a FCFS Policy 89
2.5.1 First-Come-First-Serve (FCFS) Policy................................ 89
2.5.2 Role of Dynamic Priorities, Inventory Rationing, and Price Discounts... 90
2.5.3. Numerical Insights.. 92
2.6 Conclusions 101
2.7 References.. 103

3. Optimizing Multi-Class Deterministic Demand Fulfillment through Dynamic Pricing and Inventory Rationing 107

3.1 Introduction 107

3.1.1 Problem Motivation 107

3.1.2 Literature Review 108

3.1.3 Paper Organization 110

3.2 Formulation of the Infinite Horizon Problem 111

3.3 Solution of the RD Problem 114

3.4 Role of Inventory Rationing and Price Discounting 121

3.4.1 First-Come-First-Serve (FCFS) Policy 121

3.4.2 Comparison of the RD Solution to the FCFS Solution....................122

3.4.3 Role of Price Discounting....................123

3.4.4. Static Analysis.................... 125

3.4.5. Managerial Insights and Numerical Analysis....................131

3.5 Conclusions 134

3.6 References.................... 135

Chapter 1

On the Interaction of Production and Financial Hedging Decisions in Global Markets

1.1 Introduction

1.1.1 Problem Motivation

As firms locate activities of their supply chain all over the world, and products flow cross national boundaries, managers face the uncertainties and complexities of the global environment. Exchange rates and price uncertainties in production inputs are two of the complicating factors in the global supply chain environment. Exposure to exchange rates, in particular, affects the underlying economics of any firm dealing with foreign buyers, suppliers or competitors through its impact on input costs, sales prices and volume. Such currency fluctuations can be significant (fluctuations of 1% in a day, 20% in a year are not unheard of) with drastic impact on production and sourcing costs (see Hertzell and Casper (1988), Dornier et al. (1998)).

Companies have employed different risk management approaches in coping with exchange rate and input price uncertainties. The typical way is to use financial markets, whenever possible, to hedge against such risks. Currency options are the most frequently used tools for hedging currency exposure (O'Brien (1996), Sundaram and Black (1995)). Options are financial instruments that allow a firm to buy the right, but not the obligation, to sell or buy currencies at set prices.) A sometimes overlooked option, but an effective one is for firms to use operational strategies as effective long term hedges against exchange rate and input price uncertainties. Operational hedging strategies, as clearly defined and illustrated in Cohen and Huchzermeier (1999), Cohen and Mallik (1997), and Kouvelis (1999), can be viewed as real compound options that are exercised in response to the demand, price and exchange rate contingencies faced by firms in a global supply chain context. Such real options include postponement of assembly and distribution logistics decisions, delaying final commitment of capacity and process technology investments, and/or switching production locations or sourcing partners contingent on demand and/or exchange rate scenarios. (We are going to restrict our attention to operational hedges with the real option to postpone the deployment of some of the firm resources in response to demand and exchange rate/price scenarios. For the interested reader in switching options, see work of Dixit and Pindyck (1994), Kogut and Kulatilaka (1994), Triantis et al. (1990) and Li and Kouvelis (1999)).

Even though substantial literature has been developed on both the financial hedging aspects (see Solnik (1991), O'Brien (1996), and references therein) and the operational hedging practices of price and currency risks (Cohen and Mallik (1997), Kouvelis (1999), and references therein), very little effort has been spent in developing an all encompassing risk management approach that effectively integrates financial and operational hedges. This weakness of the literature is clearly pointed out, and outlined as a future research direction, in Cohen and Huchzermeier (1999), while the potential for its effectiveness is anecdotally exposed via examples in Dornier et al. (1998, Chapter 9). Our research takes a few steps in addressing this gap in the literature. (For more detailed positioning of our research within the relevant literature, see Section 1.1.2: Literature Review).

We study the interactions of operational and financial hedging policies of risk averse global firms within a stylized, but representative, modeling setting. We consider a firm producing in its home country and selling to both the home and foreign markets. In a two stage decision framework, early capacity/production commitments and financial hedge are decided in the presence of demand and exchange rate (price) uncertainty at the first stage, while at a second stage, and after observing demand and price realizations, the firm exercises its production "allocation" option in supplying the foreign market demand (i.e., how many units to "localize" and distribute to the home and/or foreign markets). The emphasis of our analysis is on clearly establishing the value of the operational hedge ("allocation" option), and understanding how the presence of the financial hedge affects the capacities of the risk averse firm. Furthermore, interesting insights are obtained on the nature of optimal, or whenever possible perfect, financial hedges.

1.1.2 Literature Review

The literature on operational hedging practices of price and exchange rate uncertainty is rather recent, with the work of Huchzermeier and Cohen (1996) the most influential from a modeling perspective. They develop a stochastic dynamic programming formulation for the valuation of global supply chain network options with switching costs. A numerical evaluation scheme (a multinomial approximation of correlated exchange processes) is proposed for the valuation of a global supply chain network for a discrete set of alternative network options with costly switching between them. Their numerical results demonstrate the benefits of operational hedging practices via excess capacity and production switching options in environments of volatile exchange rates. The work of Kogut and Kulatilaka (1994), building upon earlier work of Dixit (1989) and Dixit and Pindyck (1994), is along the same direction with the emphasis on explicitly valuing the option of shifting production between two plants located in different countries as exchange rates fluctuate. Kouvelis, Anarloglou and Sinha (2001) study the effects of real exchange rates on the long term ownership strategies of production facilities of firms entering foreign markets, and illustrate the value of joint ventures as a "mothballing" option for such a context.

For sourcing environments with price uncertainties, some form of flexibility (i.e., in terms of the quantity and/or timing of purchase), and/or risk sharing features in the sourcing contract are often used (as documented in Carter and Vickery (1988), (1989), Dornier et al. (1998) and Tsay et al. (1999)). Li and Kouvelis (1999) explicitly study flexible and risk-sharing supply contracts under price uncertainty. Their discussion clearly illustrates how operational flexibility, supplier selection, and risk sharing, when carefully exercised, can effectively reduce the sourcing cost in environments of price uncertainty. A minimum order quantity sourcing contract in the presence of exchange rate uncertainty is analyzed in Scheller-Wolf and Tayur (1997), with their numerical results illustrating how such contracts can reduce the variance of cash flows. For a thorough coverage of the vast global supply chain literature, the detailed positioning of the operational hedging research stream within it, and other work peripherally related to it, see Cohen and Mallik (1997) and Cohen and Huchzermeier (1999).

There is a vast literature on the use of financial hedging instruments to better manage price and exchange rate uncertainties. One could start from the classical papers on option prices of Black and Scholes (1973) and Merton (1973), (1976), to more currency option specific papers of Garman and Kohlhagen (1983), Jorion (1988), Shastri and Wethyarivorn (1987), Bodurtha and Cozertadon (1987), Bigger and Hull (1983) and

Cornell and Reinganum (1991). This literature develops creative financial instruments and values them, in order to hedge uncertain magnitude cash flows, due to price/exchange rate uncertainties, but without consideration of how production decisions and operational hedging schemes, as developed in the global supply chain/operations literature, might be interacting with the magnitude and variance of such cash flows.

The international finance literature, in particular the research stream dealing with defining and measuring the different types of exchange rate exposure (see Hodder (1982), Flood and Lessard (1986)), clearly recognizes the need for a combination of financial and operational options in effectively managing operating exposure to exchange rate movements. However, beyond some intuitive and anecdotal level, discussions on the relationships between operational flexibility, financial hedging and exchange risk (see Lessard and Lightstone (1986), or a textbook level exposition at Shapiro (1988)), no structural models or quantitative tools are provided to aid the integrated operational-financial hedging decision making in uncertain price/exchange rate environments. Our research addresses this issue by clearly relating capacity/capacity choices to financial hedging strategies, and showing that for the risk averse firm the capacities it chooses are functions of both the financial hedging strategies it employs and the opportunity to exercise a real option ("allocation" option) contingent upon price and foreign market demand scenarios.

Some research papers apply the mean-variance analysis in the inventory problems. For example, Chen and Federgruen (2000) show that the risk-averse capacity is lower than the risk neutral solution in the news-vender model. On the other hand, Van Mieghem (2003) illustrates that the operational hedging may increase the capacity in the news-vender network. Different from our work, they do not consider the exchange rate uncertainty as well as the financial hedging policy.

The work of Mello, Parsons and Triantis (1995) is the one closest in spirit to our research. They present an integrated model of a multinational firm with flexibility in sourcing its production (i.e. a switching option in sourcing from different countries) and with the use of financial markets to hedge exchange rate risk. Agency costs generated by the firm's capital structure create a link between the firm's financial policy and its sourcing decisions under exchange rate uncertainty. The emphasis of the Mello et al.

paper is on valuing dynamically the operating exposure to exchange rate movements via a state-contingent model in continuous time, with the model explicitly accounting for the strategic exercise of the switching option (sourcing flexibility) of the firm. Numerical results illustrate the interdependency of sourcing flexibility and hedging strategy. Chowdhry and Howe (1999) examines a setting in which there is also uncertainty regarding the production flexibility given a fixed total production capacity. They determine the hedging policy of foreign currency cash flow and the rationale of capacity allocation when the risk-premium of financial hedge is zero. Our research focus is very different, with an emphasis on understanding the production planning implications of the opportunity to both hedge operationally (via an "allocation" option) and financially (via currency option contracts). Our stylized model allows us to obtain closed form formulas for both capacities and currency contract parameters, and thus to quantify, and structurally understand the nature and factors affecting, the magnitude of the interaction of operational and financial hedging in an uncertain price and demand environment.

1.1.3 Paper Organization

The structure of our paper is as follows: In Section 1.2, we provide our modeling framework of production and financial hedging decisions. In Section 1.3, we analyze the simultaneous production planning and financial hedging problem for a stochastic demand foreign market, and clearly explain the role of operational and financial hedging in such a context. Building on the intuition of these results, we proceed to analyze the same problem for a stochastic demand domestic market and a stochastic demand foreign market in Section 1.4. We also consider the two-market problem with a foreign production center as well as the impact of financial hedging and operational hedging on the production location in Section 1.5. We conclude with variants and extensions of our model, and a summary of our main insights, in Section 1.6.

1.2 Model

We analyze the production and financial hedging decisions of a global firm producing in a single production facility and selling to two markets: market 1, its "domestic" market (defined for our purpose as the market trading in the "home country" currency, i.e., the currency the firm uses to report its consolidated financial statements), and market 2, a "foreign country" market with uncertain currency exchange rate. The production facility could be located in either of the two markets. We start our analysis from the perceived default case of the production facility located in the "home country". We will explore the implications of the alternate location later.

We use a two-stage stochastic programming setting for modeling the firm's decisions. In the first stage, a "capacity-production" plan for the production facility is developed, and appropriate financial hedging contracts on the foreign currency are purchased in the presence of uncertainty in market demands, exchange rate, or both. We will interpret the implementation of the optimal hedging later.

In the second stage, after observing the demand and exchange rate realization, the firm makes production "allocation" decisions (e.g., how many units to appropriately configure-"localize"-and distribute) in each market so as to optimize its profits. During the first stage, the firm invests in needed technology, equipment, and factory space, or modifies existing facilities, in anticipation of market needs. With the capacities in place, commitment of production resources as part of a first stage capacity may occur, frequently in an effort to provide quick response to foreign market demand by executing prior to demand realization long lead time production activities (such as acquisition of raw materials, production of complex components and subassemblies, or even non-market specific-"vanilla"-final products). At the second stage the products are configured to the market needs, and the necessary distribution and logistic costs for supplying the markets are incurred.

We use the following notation:

s : foreign market currency exchange rate;

p_i : the revenue per unit sold in market i ;

τ_i: relevant localization costs for markets i per unit shipped (in home currency), $i = 1, 2$;

$r_1 = p_1 - \tau_1$: incremental profit per unit sold in the second stage in market 1;

$r_2(s) = sp_2 - \tau_2$: incremental profit per unit sold in the second stage in market 2;

X : capacity reserved in stage 1;

c: unit capacity reservation cost in stage 1.

d_i: random variables that represent the demand in market i, $i = 1, 2$;

$e(\cdot)$: the density function of exchange rate distribution;

$g(\cdot)$: the density function of demand distribution;

$f(\cdot,\cdot)$: the density function of joint distribution.

Given a capacity reservation X in stage 1, the stage 2 incremental profit is:

$$\pi^{op}(X, s, d_1, d_2) = \begin{cases} r_1 min\left[(X - d_2)^+, d_1\right] + r_2(s) min(X, d_2) & \text{if } r_2(s) \geq r_1 \\ r_1 min(X, d_1) + r_2(s) min\left[(X - d_1)^+, d_2\right] & \text{if } r_1 > r_2(s) \geq 0 \\ r_1 min(X, d_1) & \text{if } r_2(s) < 0 \end{cases} \quad (1.1)$$

where $x^+ = max(x, 0)$. As (1.1) suggests, when $r_2(s) \geq r_1$, we first allocate produced units to meet demand in market 2, and the remaining units, if any, are localized for the needs of market 1. The priorities are reversed in the allocation process when $r_1 > r_2(s) \geq 0$. For the case $r_2(s) < 0$, we only localize the produced units for market 1 needs.

Let h denote a generic financial hedging contract and $R_h(s)$ denote the payoff of h in stage 2. Thus the firm's profit at time T (the end-of-period) is given as:

$$\pi(X, h, s, d_1, d_2) = \left[-cX - H(h)\right]e^{\gamma T} + \pi^{op}(X, s, d_1, d_2) + R_h(s). \quad (1.2)$$

$$\pi(X, h, s, d_1, d_2) = \left[-cX - H(h)\right]e^{\gamma T} + \pi^{op}(X, s, d_1, d_2) + R_h(s). \quad (1.2)$$

γ is

where $H(h)$ is the cost of acquiring financial contract h incurred in stage 1, and γ is wise explicitly noted, that the firm is risk-averse. Furthermore, we assume that

(a) the firm's objective is to maximize the expected utility for time T profit, and

(b) the firm's expected utility is represented by

$$U(\pi)=E[\pi]-\lambda V[\pi]$$

where $E[\cdot]$ and $V[\cdot]$ are the expected value and the variance of time T payoff, respectively, and $\lambda \geq 0$ represents the rate at which the firm will substitute variance for expected value. It is well known (see Phillippatos and Gressis (1975), Jucker and Carlson (1976)) that the use of the mean-variance criterion is consistent with the principle of maximizing expected utility if

(i) the firm's utility function can be represented by a quadratic function of time T payoff, or

(ii) the probability distribution of time T payoff is a two parameter distribution (e.g. normal distribution).

In this paper we use assumption (i). Justification for the use of total risk measure (such as $V[\cdot]$) instead of measures of systematic risk is provided in Hodder and Dincer (1986) with the main argument being that managers are typically concerned about total risk. This can be attributed to a concern with the probability of financial distress, or bankruptcy, as well as to agency considerations (see Jensen and Meckling (1976), Markus (1982)). Therefore, the firm's production and financial hedging problem is

$$\max_{X\geq 0, h\in\Omega}\left\{E\left[\pi(X,h,s,d_1,d_2)\right]-\lambda V\left[\pi(X,h,s,d_1,d_2)\right]\right\} \tag{1.3}$$

where Ω is a given feasible financial hedging set.

In pursuit of a solution to (1.3), we need to first understand how we can financially hedge a given capacity X. For doing that, we need to first describe the type of financial contracts we are considering, and how these can be valued.

We are considering currency option contracts. We will do most of our analysis with call option currency contracts, but we will later comment on features of put option contracts if used instead. Let us consider a simple call option currency contract $h=(Q,S)$, where S is the exercise price of the call option, and Q is the contract size. The payoff of the call option in the second stage is

$$R_h(s)=(s-S)^+Q.$$

Let $C(S)$ denote the price of per unit call option contract in the first stage, which can be observed in the exchange rate market or some financial theory. For example, if the currency exchange rate *s(T)* follows lognormal distribution, $C(S)$ can be determined by using Black-Schole's valuation (see Black and Schole (1973)). The risk premium of per unit call option contract is defined as

$$\Delta C(S) = e^{\gamma T} C(S) - E\left[(s-S)^{+}\right]$$

In the profit expression of our model as in (1.1)-(1.3), we have included beyond the financial hedge related revenue terms, capacity acquisition costs in the first stage, and sales revenue and incremental variable costs (for localizing and distributing items) at the second stage. For some environments, it might be appropriate to incorporate shortage penalties, for when the foreign market demand is not met in full (e.g., goodwill loses with distribution channels and customers, potential implications for future sales etc.) and excess capacity costs associated with the unused capacity committed to the foreign market. (For detail motivation of such costs, see Silver, Pyke and Peterson (1998)).

Let v_{p1} and v_{p2} be the shortage penalty cost per unit of unmet domestic and foreign market demand, respectively; and v_s be the salvage value per unit of excess capacity at the second stage (assumed $v_s < ce^{\gamma T}$).

Also, define $\tau_1' = \tau_1 - v_{p1} + v_s$, $\tau_2' = \tau_2 - v_{p2} + v_s$, $r_1' = p_1 - \tau_1'$ and $r_2' = p_2 s - \tau_2'$. Then our incremental profit at the second stage is now

$$\pi^{op'}(X,s,d_1,d_2)=\begin{cases}\begin{pmatrix} r_1' min\left[(X-d_2)^+,d_1\right]+r_2' min(X,d_2) \\ v_s(X-d_1-d_2)^+ - v_{p1}\left(d_1 - min\left[(X-d_2)^+,d_1\right]\right) \\ -v_{p2}(d_2 - min(X,d_2)) \end{pmatrix} & \text{if } r_2' \geq r_1' \\ \begin{pmatrix} r_1' min(X,d_1)+r_2' min\left[(X-d_1)^+,d_2\right] \\ v_s(X-d_1-d_2)^+ - v_{p1}(d_1-min(X,d_1)) \\ -v_{p2}\left(d_2 - min\left[(X-d_1)^+,d_2\right]\right) \end{pmatrix} & \text{if } r_1' > r_2' \geq 0 \\ \begin{pmatrix} r_1' min(X,d_1)+v_s(X-d_1)^+ \\ -v_{p1}(d_1-min(X,d_1)) - v_{p2}d_2 \end{pmatrix} & \text{if } r_2' < 0 \end{cases}$$

Observe that

$$\pi^{op'}(X,s,d_1,d_2)=\pi^{op}\Big|_{\tau_1',\tau_2',r_1',r_2'}(X,s,d_1,d_2)+v_sX-v_{p1}d_1-v_{p2}d_2$$

where $\pi^{op}\big|_{\tau_1',\tau_2',r_1',r_2'}$ is the profit function in (1.1) adjusted the parameter set as $\left(\tau_1',\tau_2',r_1',r_2'\right)$. Comparing with (1.1), we see that our second stage incremental profit has the "added component" of $v_sX-v_{p1}d_1-v_{p2}d_2$. The costs from such added component can be easily incorporated to the model, without affecting the methodological aspects and/or the qualitative aspects of our analysis. They do, however, complicate already complex formulas and proofs, without adding intuition and/or insights to the ones provided by our original model (as in (1.1)-(1.3)). Thus, in the rest of the paper, we focus on the problem without shortage penalty and salvage value unless explicitly addressed.

1.3 Hedging the Currency Risk of a Stochastic Demand Foreign Market

1.3.1 Problem Solution

We consider the problem with only one foreign market (i.e. $d_1 = 0$) in this section. The profit function in (1.1) is simplified as

$$\pi^{op}(X,s,d_2) = \begin{cases} r_2(s)\min(X,d_2) & \text{if } r_2(s) \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

We omit the market index $i = 2$ from now until the end of this section since there is only one market. We use in this section the notational convention that all expectation operators, if not otherwise noted, are over the joint distribution of the exchange rate and demand random variables. Let $\widehat{S} = (S_1,\ldots,S_n)$ be a given exercise price vector of call options, respectively. (The symbol $\widehat{\ }$ on the top of variables is a reminder that the variables in this expression are vectors). The appropriated revenue and price vector are

$$\widehat{R}_h(s) \equiv \left[(s-S_1)^+ Q_1,\cdots,(s-S_n)^+ Q_n\right] \text{ and } \widehat{C}(\widehat{S}) \equiv \left[C(S_1),\cdots,C(S_n)\right]$$

Proposition 1.1. *For a stochastic demand foreign market, given a capacity X and using a group of call option currency contracts at a prespecified exercise price vector $\widehat{S}$, the optimal hedge size vector $\widehat{Q}(X,\widehat{S})$ is the unique solution to the linear equations*

$$Cov\left[\left(s-\widehat{S}\right)^+,\left(s-\widehat{S}\right)^+\right]\widehat{Q}^\wedge = -Cov\left(\pi^{op}(X,s,d),\left(s-\widehat{S}\right)^+\right) - \left[\Delta\widehat{C}(\widehat{S})\right]^\wedge / 2\lambda \quad (1.4)$$

where $Cov(x,y)$ is the covariance matrix of x and y, $x^\wedge$ is the transpose of x, and

$$\Delta\widehat{C}\left(\widehat{S}\right) = \left(e^{\gamma T}C(S_1) - E\left[(s-S_1)^+\right],\ldots,e^{\gamma T}C(S_n) - E\left[(s-S_n)^+\right]\right)$$

and $\left(s-\widehat{S}\right)^+ = \left((s-S_1)^+,\ldots,(s-S_n)^+\right)$.

Proof: We obtain (1.4) from the first order condition on the hedge sizes. The full rank of $Cov\left[\left(s-\widehat{S}\right)^{+},\left(s-\widehat{S}\right)^{+}\right]$ results the uniqueness of the solution. ■

For a given demand distribution, it may be not difficult to choose reasonable exercise price vector to obtain optimal or efficient financial hedging policy.

Example 1.3.1. Assume that $\ln(s)\sim N(\mu,\sigma)$, $d=(a-bs)^{+}$, where μ,σ,a,b are constants. Given a capacity X, the number of units allocated to the foreign market is

$$Y(s)=\begin{cases} min\left(X,(a-bs)^{+}\right) & r\geq 0 \\ 0 & \text{Otherwise} \end{cases}.$$

If $b=0$, the hedge policy $\left(S^{0},Q^{0}\right)=(\tau/p,-pmin(X,a))$ results the zero profit variance . Moreover, the hedging policy consisting of $(\tau/p,Q)$, where Q are the solution from (1.4), reduces most of the profit variance if b is small.

If b is large, we introduce a complex optimal hedge policy. Given a group of call option exercise price vector as $\widehat{S}=(\tau/p,\tau/p+\Delta,\cdots,\tau/p+n\Delta)$, where Δ is a small increment, we choose $Q_i=Y(S_{i-1})-Y(S_i)$, where $i=1,\cdots,n$ and given $Y(S_{n+1})=0$. It is easy to verify that this hedge policy is close to a zero-variance hedge if $\Delta\to 0$ and $n\Delta\to\infty$.

If the option risk premium is small in the exchange rate market (see Bigger and Hull 1983), i.e. $\Delta\widehat{C}\left(\widehat{S}\right)\to 0$, then the hedge policy almost does not effect the expected profit and the zero-variance hedge is optimal. ■

The single call option hedging in the following corollary is a natural choice for many practical problems, since we allocate the units to the foreign market only if $s\geq\tau/p$ at stage 2.

Corollary of Proposition 1.1. *For a stochastic demand foreign market, given a capacity* X *, the optimal hedge size for a single call option with* $S = \tau/p$ *is*

$$Q(X,\tau/p) = -\frac{pCov\left[r^+ min(X,d), r^+\right]}{V\left[r^+\right]} - \frac{p^2\Delta C(\tau/p)}{2\lambda V\left[r^+\right]} \qquad (1.5\text{-}a)$$

For the more general problem with shortage penalty and salvage value, the optimal hedge size for $S = \tau/p$ *is*

$$Q'\left(X,\frac{\tau'}{p}\right) = -\frac{pCov\left[\left(r'\right)^+ min(X,d), \left(r'\right)^+\right]}{V\left[\left(r'\right)^+\right]} - \frac{p^2\Delta C(S)}{2\lambda V\left[\left(r'\right)^+\right]} + \frac{pCov\left[v_p d, \left(r'\right)^+\right]}{V\left[\left(r'\right)^+\right]} \qquad (1.5\text{-}b)$$

Proof: Similar to Proposition 1.1. ■

Comparing with (1.5-a) and (1.5-b), it is apparent that the optimal hedge size now has an added term, which is used to hedge "the added component" in the incremental second stage profit.

The strong results are presented for a stochastic demand foreign market with the demand and exchange rate being independent random variables, referred as the *independent case*.

Proposition 1.2. *For a stochastic demand foreign market and a risk averse firm, with the demand and exchange rate being independent random variables, given* $S^0 = \tau/p$*, the production-financial hedging decisions optimizing the mean-variance objective are a pair of unique value, denoted as* $\left(X^0, Q^0 \middle| S^0\right)$*, such that:* X^0 *is the solution to the equation*

$$\left[E\left(r^+\right) + \Delta C(S^0)\right]G^0(X) = ce^{\gamma T} + 2\lambda E\left[\left(r^+\right)^2\right]\left[X + G^1(X) - G^1(0)\right]G^0(X) \qquad (1.6\text{-}a)$$

and

$$Q^0 = pG^1\left(X^0\right) - pG^1(0) - p^2\Delta C(S^0)/2\lambda V\left[r^+\right]. \qquad (1.6\text{-}b)$$

Proof: Given X and a single call option $Q=Q(X,\tau/p)$ and $S=\tau/p$, we need to calculate

$$\frac{\partial\left(E\left[\pi(X,h,s,d)\right]-\lambda V\left[\pi(X,h,s,d)\right]\right)}{\partial X}$$

Observe that for the independent case

$$\frac{\partial Q(X,\tau/p)}{\partial X}=-p\int_X^{\infty}g(d)dd\,,$$

$$\frac{\partial\left(E\left[\pi(X,h,s,d)\right]\right)}{\partial X}=\left[E\left(r^+\right)+\Delta C(\tau/p)\right]\cdot\int_X^{\infty}g(d)dd-ce^{\gamma T}.$$

Define $A(X)=\pi^{op}(X,s,d)-E\left[\pi^{op}(X,s,d)\right]+Q\left((s-\tau/p)^+-E\left[(s-\tau/p)^+\right]\right)$, we have

$$\frac{\partial V\left[\pi(X,h,s,d)\right]}{\partial X}=2E\left(A(X)\frac{\partial\left[\pi^{op}(X,s,d)\right]}{\partial X}\right)-2E\left(A(X)\frac{\partial E\left[\pi^{op}(X,s,d)\right]}{\partial X}\right)$$
$$+2E\left(A(X)\frac{\partial\left[Q\left((s-\tau/p)^+-E\left[(s-\tau/p)^+\right]\right)\right]}{\partial X}\right)$$

Calculate these terms separately

$$E\left(A(X)\frac{\partial\left[\pi^{op}(X,s,d)\right]}{\partial X}\right)$$
$$=\int_X^{\infty}\int_0^{\infty}X\left(r^+\right)^2e(s)g(d)dsdd-E\left[min(X,d)\right]\cdot\left[E\left(r^+\right)\right]^2\cdot\int_X^{\infty}g(d)dd$$
$$-\left(pE\left[min(X,d)\right]-\frac{p^2\Delta C(\tau/p)}{2\lambda V\left[r^+\right]}\right)(1/p)V\left(r^+\right)\cdot\int_X^{\infty}g(d)dd$$
$$=\left(E\left[\left(r^+\right)^2\right]\int_0^X(X-d)g(d)dd-\frac{p}{2\lambda}\Delta C(\tau/p)\right)\cdot\int_X^{\infty}g(d)dd$$

and

$$E\left(A(X)\frac{\partial E\left[\pi^{op}(X,s,d)\right]}{\partial X}\right)=E\left[A(X)\right]\cdot E\left[r^+\right]\cdot\int_X^{\infty}g(d)\cdot dd=0$$

and

$$E\left(A(X)\frac{\partial\left[Q\left((s-\tau/p)^{+}-E\left[(s-\tau/p)^{+}\right]\right)\right]}{\partial X}\right)$$

$$=\left[-\int_X^{\infty} g(d)dd\right]\cdot E\left[A(X)\cdot\left(r^{+}-E\left[r^{+}\right]\right)\right]$$

$$=\left[-\int_X^{\infty} g(d)dd\right]\cdot V\left[r^{+}\right]\cdot\left(E[min(X,d)]-\left(E\left[min(X,d)\right]+\frac{p\Delta C(\tau/p)}{2\lambda V\left[r^{+}\right]}\right)\right)$$

$$=\frac{p\Delta C(\tau/p)}{2\lambda}\int_X^{\infty} g(d)\cdot dd$$

Plugging in the above three equations result

$$\frac{\partial V\left[\pi(X,h,s,d)\right]}{\partial X}=2E\left[\left(r^{+}\right)^{2}\right]\cdot\int_0^X (X-d)\,g(d)dd\cdot\int_X^{\infty} g(d)dd\,.$$

The first order condition results

$$\left(E\left(r^{+}\right)+\Delta C(\tau/p)+2\lambda E\left[\left(r^{+}\right)^{2}\right]\cdot\left[-\int_0^X (X-d)\,g(d)dd\right]\right)\cdot\int_X^{\infty} g(d)\cdot dd=ce^{\gamma T}$$

Furthermore, observe that both $-\int_0^X (X-d)\,g(d)dd$ and $\int_X^{\infty} g(d)\cdot dd$ are decreasing as X increases, which are sufficient to prove unique optimality, since the right side of the above equation is decreasing in X if it is positive and there exists one and only one solution can satisfy the first order condition. Thus, we obtain (1.6-a) and (1.6-b) follows from (1.5-a) directly ■

The term $-p^{2}\Delta C(\tau/p)/2\lambda V\left[r^{+}\right]$, called the *hedge size deviation*, is independent with demand distribution and capacity. It is a trade-off between the mean and variance only dependent on the risk premium, which exists in the financial market even if the capacity and demand are zero. In the exchange rate market, the risk premium of call option is typically very small (see Bigger and Hull (1983)). The firm cannot make much extra money from selling a call option and the hedge size deviation is small if λ is in a reasonable region.

Example 1.3.2. Let the exchange rate follow a log normal distribution with $\ln(s) \sim N(1,1)$. The demand is independent with exchange rate and follows a normal distribution $E(d) = 100$ and $\sigma(d) = 30$. Let also $ce^{\gamma T} = 2$, $p = 1$, $\tau = 1.5$ and $\lambda = 0.0002$. Define the *risk rate* as $\Delta C(\tau/p)/C(\tau/p)$. We present the results for the cases whether or not considering risk premium or financial hedge in the following table. ■

$\frac{\Delta C(\tau/p)}{C(\tau/p)}$		(1) Ignore Risk Premium	(2) Optimal Hedging	$\frac{(2)-(1)}{(2)}(\%)$
−0.01	X^*, U^* $E^*, \lambda V^*$	87.6953,77.7115 79.2093,1.4978	87.3047,76.0625 77.6132,1.5507	−0.45,−2.12 -2.02,3.53
0.00	X^*, U^* $E^*, \lambda V^*$	87.6953,77.7115 79.2093,1.4978	87.6953,77.7115 79.2093,1.4978	0,0 0,0
0.01	X^*, U^* $E^*, \lambda V^*$	87.6953,77.7115 79.2093,1.4978	88.2813,79.6318 81.2546,1.6228	0.67,2.47 2.58,8.35
0.02	X^*, U^* $E^*, \lambda V^*$	87.6953,77.7115 79.2093,1.4978	88.6719,81.0464 82.9418,1.8954	1.11,4.29 4.71,26.55

Table 1.1. The Effect of Risk Premium

Table 1.1 illustrates that the relative error of capacity and the absolute error of variance are very small. In the practical environments, if there exists the risk premium, it affects the profit no matter which kind of capacity we choose, the optimal one or the one ignore the risk premium. The real errors on expected profits and objective values are even smaller than the data in Table 1. This implies that the effect of risk premium is minor.

When the risk premium is zero, the financial hedging does not affect the expected profit and the optimal hedging policy is the one of minimizing the profit variance. Thus, we neglect risk premium and present the strong results in the following subsection.

1.3.2 Solution of the Problem with Zero Risk Premium

Assuming the risk premium is zero in this subsection, i.e. $\Delta C(S)=0$, we can find the optimal financial hedging policy among the combination of call options, put options and forward contracts.

Let $h^p=(Q^p,S^p)$ denote a simple put option currency contract. The payoff at stage 2 is $R_{h^p}(s)=(S^p-s)^+Q^p$. Let $h^f=(Q^f)$ denote a forward contract. The payoff at stage 2 is $R_{h^f}(s)=(s-E(s))Q^f$. (We use the superscript " p " and " f " to identify the variables related to the put option and forward contract, respectively.)

Noting that, $R_{h^f}(s)$ is the same as the payoff of call option $h=(Q^f,0)$ as well as the put option $h^p=(Q^f,\infty)$ at stage 2. These three hedging policies can be replicated with each other and their effects on the profit mean and variance are the same. Thus, we treat these three financial contracts as equivalent in this paper. (We use "remarks" instead of "propositions" to identify the results under zero-risk-premium assumption and present the strong results for the independent case as follows.)

Remark 1.1. *For a stochastic demand foreign market, with zero risk premium and the demand and exchange rate being independent random variables, given a capacity X and an exercise price S, the optimal size of single call option is*

$$Q(X,S)=-E[min(X,d)]\cdot Cov\left[r^+,(s-S)^+\right]\Big/V\left[(s-S)^+\right], \qquad (1.7\text{-}a)$$

the optimal size of forward contract is

$$Q^f(X)=-E[min(X,d)]\cdot Cov\left[r^+,s\right]\Big/V[s], \qquad (1.7\text{-}b)$$

and the optimal size of single put option is

$$Q^p(X,S)=-E[min(X,d)]\cdot Cov\left[r^+,(S-s)^+\right]\Big/V\left[(S-s)^+\right]. \qquad (1.7\text{-}c)$$

Given a capacity X, the optimal hedging policy among the combination of call options and put options and forward contracts is a single call option

$$Q^*(X)=-E\left[min(X,d)\right]=G^1(X)-G^1(0) \text{ and} \qquad (1.7\text{-}d)$$

$$S^*=\tau/p. \qquad (1.7\text{-}e)$$

Proof: Given X and S, the optimal hedging sizes of call option and put option and forward contract can be obtained from first order condition on $V\left[\pi(X,h,s,d)\right]$ similar to Proposition 1.1.

Let us consider a single put option contract $\left(Q^p\left(S^p\right),S^p\right)$ and compare it to the put option $\left(Q^p\left(\infty\right),\infty\right)$. Define the put option notation $h^p\big|_S \equiv \left(Q^p\left(S\right),S\right)$. It is easy to verify that

$$\Delta V^p\left(S^p,\infty\right) \equiv V\left[\pi\left(X,h^p\big|_{S^p},s,d\right)\right]-V\left[\pi\left(X,h^p\big|_{\infty},s,d\right)\right]$$
$$=E\left(\left[\pi^{op}\left(X,s,d\right)+R_{h^p|_{S^p}}\left(s\right)\right]^2\right)-E\left(\left[\pi^{op}\left(X,s,d\right)+R_{h^p|_{\infty}}\left(s\right)\right]^2\right)$$
$$=\int_{\infty}^{S^p}\frac{\partial\left[E\left(\left[\pi^{op}\left(X,s,d\right)+R_{h^p|_S}\left(s\right)\right]^2\right)\right]}{\partial S}dS$$

Define a hedging policy $\bar{h}^p\big|_S \equiv \left(\bar{Q}^p\left(S\right),S\right)$, where $\bar{Q}^p(S)=Q^p(S)/E\left[min\left(X,d\right)\right]$ is the optimal hedge size of the simplified problem with $X=\bar{d}=1$. Then

$$\Delta V^p\left(S^p,\infty\right)=\left(E\left[min\left(X,d\right)\right]\right)^2\cdot\int_{\infty}^{S^p}\frac{\partial\left[E\left(\left[\pi^{op}\left(1,s,1\right)+R_{\bar{h}^p|_S}\left(s\right)\right]^2\right)\right]}{\partial S}dS$$
$$=\left(E\left[min\left(X,d\right)\right]\right)^2\cdot\left(V\left[\pi\left(1,\bar{h}^p\big|_{S^p},s,1\right)\right]-V\left[\pi\left(1,\bar{h}^p\big|_{\infty},s,1\right)\right]\right)$$

Similarly, define the notations $h\big|_S \equiv \left(Q\left(S\right),S\right)$ and $\bar{h}\big|_S$ for the single call option, compare $h\big|_{\tau/p}$ and $h\big|_0$, and observe that

$$\Delta V\left(\tau/p,0\right) \equiv V\left[\pi\left(X,h\big|_{\tau/p},s,d\right)\right]-V\left[\pi\left(X,h\big|_0,s,d\right)\right]$$
$$=\left(E\left[min\left(X,d\right)\right]\right)^2\cdot\left(V\left[\pi\left(1,\bar{h}\big|_{\tau/p},s,1\right)\right]-V\left[\pi\left(1,\bar{h}\big|_0,s,1\right)\right]\right)$$

The hedge effect of policy $\bar{h}\big|_0$ is equivalent to $\bar{h}^p\big|_{\infty}$ by verifying

$$V\left[\pi\left(1,\bar{h}\big|_0,s,1\right)\right]=V\left[\pi\left(1,\bar{h}^p\big|_{\infty},s,1\right)\right]$$

Observing $V\left[\pi\left(1,\bar{h}\Big|_{\tau/p},s,1\right)\right]=0$, we obtain

$$V\left[\pi\left(X,h^p\Big|_{S^p},s,d\right)\right]-V\left[\pi\left(X,h\Big|_{\tau/p},s,d\right)\right]$$
$$=\Delta V^p\left(S^p,\infty\right)-\Delta V\left(\tau/p,0\right)=\left(E[min(X,d)]\right)^2\cdot V\left[\pi\left(1,\bar{h}^p\Big|_{S^p},s,1\right)\right]\geq 0$$

and result the optimal hedging policy among the single call and put options.

The optimality can be generalized by the similar approach to the range among all combinations of call and put options and forward contracts. ■

Some managers may prefer to the forward hedging policy by the practical transaction cost. We provide the capacity applying forward hedging in the following corollary.

Corollary of Proposition 1.2. *For a stochastic demand foreign market, with the zero risk premium and the demand and exchange rate being independent random variables, applying the forward hedging policy, the optimal capacity* X^f *is uniquely determined by*

$$\begin{pmatrix} E\left(r^+\right)-2\lambda E\left[\left(r^+\right)^2\right]\int_0^X (X-d)\,g(d)dd \\ -2\lambda\left[1-\left(Cov\left(r^+,s\right)/V[s]\right)^2\right]V[s]E[min(X,d)] \end{pmatrix}\int_X^\infty g(d)dd=ce^{\gamma T} \qquad (1.8)$$

Proof: We need to calculate

$$\frac{\partial\left(E\left[\pi\left(X,h^f,s,d\right)\right]-\lambda V\left[\pi\left(X,h^f,s,d\right)\right]\right)}{\partial X}$$

for $Q^f(X)=-\alpha E[min(X,d)]$, where $\alpha=Cov\left[r^+,s\right]/V[s]$, we have

$$\frac{\partial Q^f(X)}{\partial X}=-\alpha\int_X^\infty g(d)\cdot dd$$

$$\frac{\partial(E[\pi])}{\partial X}=E\left(r^+\right)\cdot\int_X^\infty g(d)\cdot dd-ce^{\gamma T}$$

and
$$\frac{\partial V\left[\pi(X,h^f,s,d)\right]}{\partial X}=E\left[\left(r^+\right)^2\right]\cdot\int_0^X (X-d)\,g(d)dd\cdot\int_X^\infty g(d)dd$$
$$+\left(1-\alpha^2\right)\cdot V[s]\cdot E[min(X,d)]\cdot\int_X^\infty g(d)dd$$

Plugging in the above equations result

$$\frac{\partial\left(E\left[\pi(X,h,s,d)\right]-V\left[\pi(X,h,s,d)\right]\right)}{\partial X}$$
$$=\begin{pmatrix} E\left(r^+\right)+2\lambda E\left[\left(r^+\right)^2\right]\cdot\left[-\int_0^X (X-d)\,g(d)dd\right] \\ +2\lambda\left[\left(Cov\left(r^+,s\right)/V[s]\right)^2-1\right]\cdot V[s]\cdot E[min(X,d)] \end{pmatrix}\cdot\int_X^\infty g(d)dd-ce^{\gamma T}$$

■

When τ is small, we have $Cov\left(r^+,s\right)$ is close to $V[s]$ and the two considered capacities $\left(X^f,X^0\right)$ are close to each other from (1.6a) and (1.8).

For the problem with zero risk premium and zero mean-variance ratio, i.e. $\lambda=0$, we focus on find a capacity to maximize the expected profit and ignore the financial hedging. This problem is the traditional "risk-neutral" problem. (We use the subscript N as a reminder that the variables in this expression are related to the risk-neutral problem).

Remark 1.2. *For a stochastic demand foreign market with zero risk premium, the optimal risk-neutral capacity* X_N^* *is non-zero (and unique) if* $E\left[r^+\right]>c$ *(and the joint distribution* $f(s,d)>0$ *for all* s *and* d *realizations) and solves*

$$\int_X^\infty\int_{\tau/p}^\infty rf(s,d)dsdd=ce^{\gamma T}. \qquad (1.9\text{-}a)$$

Otherwise, both the optimal production quantity and the optimal expected profit are zero.

For the case of demand and exchange rate being independent random variables, if $E\left[r^+\right]>c$, *then* X_N^* *is given by a newsboy-like formula*

$$G^0\left(X_N^*\right)=ce^{\gamma T}/E\left(r^+\right). \qquad (1.9\text{-}b)$$

For the case of fixed demand, denoted as $\bar{d}$, *we have*

$$X_N^* = \bar{d} \text{ if } E\left[r^+\right] > ce^{\gamma T}, \text{ otherwise, } X_N^* = 0. \qquad (1.9\text{-}c)$$

Proof: Observe that the first order derivative of $E\left[\pi\left(X,h,s,d\right)\right]$ is

$$F(X) = -ce^{\gamma T} + \int_X^{\infty}\int_{\tau/p}^{\infty} rf(s,d)dsdd \text{ , furthermore, } F(0) = -ce^{\gamma T} + E\left[r^+\right].$$

For the case $E\left[r^+\right] \le c$, the optimal capacity $X_N^* = 0$ since $F(X) \le 0$ for any $X \ge 0$. For the case $E\left[r^+\right] > c$, $F(X)$ decreases in X. If $f(s,d)$ is strictly positive for the whole domain of its definition, then $F(X)$ strictly decreases and the solution for $F(X_N^*) = 0$ is unique. Otherwise the solution for $F(X_N^*) = 0$ may be a continuous interval. The formula for the independent case and fixed demand follow by simple algebra. ■

1.3.3 Role of Production and Financial Hedges

When a firm exploits the foreign market or the foreign production center, typically, the advantage is the higher selling price or the lower production cost and the disadvantage is the higher risk. In this paper, we try to use the operational hedge and financial hedge to reduce the risk (measured by the profit variance) and improve the expected profit. The operational and financial hedge opportunities are contingent on the exchange rate uncertainty. The hedging benefits will be vanished if the exchange rate is a constant.

We will first study the impact of an allocation option for a risk-averse firm. We will start with the case that no financial hedge is used. The mean-variance objective in this case is

$$\begin{aligned} &E\left[\pi^{op}\left(X,s,d\right)\right] - \lambda V\left[\pi^{op}\left(X,s,d\right)\right] \\ &= -cXe^{\gamma T} + E\left[r^+ min(X,d)\right] - \lambda V\left[r^+ min(X,d)\right] \end{aligned} \qquad (1.10)$$

The first order derivative function of this mean-variance objective is

$$F_{A,-H}(X) = -ce^{\gamma T} + \int_0^{\infty}\int_X^{\infty} r^+ f(s,d)ddds - 2\lambda X \int_0^{\infty}\int_X^{\infty} (r^+)^2 f(s,d)ddds \\ + 2\lambda E\left[r^+ min(X,d)\right]\int_0^{\infty}\int_X^{\infty} r^+ f(s,d)ddds \tag{1.11}$$

(The subscripts A and $-A$ are a reminder that the variables in this expression are related to the policy with operational hedging and without operational hedging, respectively. So are the subscripts H and $-H$ for financial hedge).

The first order derivative function without an allocation option (and without a financial hedge) is

$$F_{-A,-H}(X) = -ce^{\gamma T} + \int_0^{\infty}\int_X^{\infty} rf(s,d)\cdot dd\cdot ds - 2\lambda X \int_0^{\infty}\int_X^{\infty} r^2 f(s,d)\cdot dd\cdot ds \\ + 2\lambda E\left[rmin(X,d)\right]\cdot \int_0^{\infty}\int_X^{\infty} rf(s,d)\cdot dd\cdot ds \tag{1.12}$$

From (1.11) and (1.12), we can establish that

$$F_{A,-H}(X) - F_{-A,-H}(X) = F_e(X) + 2\lambda F_v(X) \tag{1.13}$$

where

$$F_e(X) = \int_0^{\tau/p}\int_X^{\infty} |r| f(s,d)\cdot dd\cdot ds$$

and

$$F_v(X) = X\int_0^{\tau/p}\int_X^{\infty} r^2 f(s,d)ddds + E\left[r^+ min(X,d)\right]\int_0^{\infty}\int_X^{\infty} r^+ f(s,d)ddds \\ - E\left[rmin(X,d)\right]\int_0^{\infty}\int_X^{\infty} rf(s,d)ddds$$

and $|x|$ is the absolute value of x.

The terms $F_e(X)$ and $F_v(X)$ represent the impact of the allocation option on the expected profit and variance of the expected profit, respectively.

For the case with $F_e(X) + 2\lambda F_v(X) \geq 0$, we can conclude that the optimal capacity with an allocation option, $X^*_{A,-H}$, is at least as large as that without an allocation plan, $X^*_{-A,-H}$, for a risk averse firm not using a financial hedge. Furthermore the allocation option increases the expected profit and reduces the variance of it.

On the other hand, in some cases the allocation option might result in an increased profit variance. Then, we might obtain $X^*_{A,-H} \leq X^*_{-A,-H}$. In such a case, the firm can increases the mean-variance objective value by using the capacity $X^*_{-A,-H}$ but implementing it with an allocation option at the second stage.

We state below a stronger result for the independent case on the impact of the allocation option when there is no financial hedge used.

Proposition 1.3. *For a stochastic demand foreign market with the demand and exchange rate being independent random variables, a risk averse firm not hedging financially can increase the capacity, improve its expected profit and therefore improve upon its mean-variance objective, by implementing an allocation option.*

Proof: The first order derivative of the mean-variance objective with an allocation option in this case is

$$F_{A,-H}(X) = -ce^{\gamma T} + \begin{pmatrix} E\left[r^+\right] - 2\lambda V\left[r^+\right]E[\min(X,d)] \\ -2\lambda E\left[\left(r^+\right)^2\right]\int_0^X (X-d)\,g(d)dd \end{pmatrix} \int_X^\infty g(d)dd$$

Note that at the optimal capacity $X^*_{A,-H}$, the function $F_{A,-H}(X)+ce^{\gamma T}$ is equal to $ce^{\gamma T} \geq 0$. Since $F_{A,-H}(X)+ce^{\gamma T}$ is the product of two decreasing functions in X and $\int_X^\infty g(d)dd$ is positive, the optimal capacity satisfying the above condition is unique. Following the similar analysis, we can state that $X^*_{-A,-H}$ is unique, and the corresponding first order derivative function is

$$F_{-A,-H}(X) = -ce^{\gamma T} + \begin{pmatrix} E(r) - 2\lambda V(r)E[\min(X,d)] \\ -2\lambda E(r^2)\int_0^X (X-d)\,g(d)dd \end{pmatrix} \int_X^\infty g(d)dd$$

Thus, the fact $F_{A,-H}(X) \geq F_{-A,-H}(X)$ implies the results in Proposition 1.3. ■

The use of a financial hedge achieves its obvious intent to reduce the variance of profits given capacity, and consequently increase the mean-variance objective value. The discussion for the independent case follows.

Proposition 1.4. *For a stochastic demand foreign market with the demand and exchange rate being independent random variables, the use of an single call option hedging given exercise price* $S = \tau/p$ *(or a forward hedging) increases the capacity, the expected profit and the objective value when the firm using (or no using) an allocation option if the risk premium is positive.*

Proof: For the independent case the firm uses an allocation option, we choose the financial hedge policy as a call option

$$Q(X,\tau/p) = -pE\left[\min(X,d)\right] - \frac{p^2 \Delta C(\tau/p)}{2\lambda V\left[r^+\right]} \text{ and } S = \tau/p.$$

The first order derivative of the mean-variance objective leads to

$$F_{A,H}(X) = -ce^{-\gamma T} + \begin{bmatrix} E\left(r^+\right) + \Delta C(\tau/p) \\ -2\lambda E\left[\left(r^+\right)^2\right]\int_0^X (X-d)\,g(d)dd \end{bmatrix} \int_X^\infty g(d)dd$$

Using similar arguments in Proposition 1.3, the optimal capacity is unique. Furthermore, to assess the impact of the financial hedge, observe that

$$F_{A,H}(X) - F_{A,-H}(X) = \left(\Delta C(\tau/p) + 2\lambda \cdot V\left[r^+\right] \cdot E\left[min(X,d)\right]\right)\int_X^\infty g(d)dd \geq 0$$

Thus, the financial hedge increases the optimal capacity, the expected profit and the objective value (through increased risk premium benefit and decreased variance of profits).

Similar observations can be made for the case when the firm no using an allocation option. We choose financial hedge of shorting a forward contract of size $\left|Q^*_{-A,H}\right| = E\left[min(X,d)\right]$ and results

$$F_{-A,H}(X) = -ce^{\gamma T} + \left(E(r) + \Delta C^f - 2\lambda E\left(r^2\right)\left[\int_0^X (X-d)\,g(d)dd\right]\right)\int_X^\infty g(d)dd$$

and $$F_{-A,H}(X) - F_{-A,-H}(X) = \Delta C^f + 2\lambda V(r) E\left[min(X,d)\right]\int_X^\infty g(d) \cdot dd \geq 0$$

where ΔC^f is the risk premium of forward contract. ■

We state below a similar results for the zero risk premium case on the impact of the allocation option when the financial hedge is used.

Remark 1.3. *For a stochastic demand foreign market with zero risk premium and the demand and exchange rate being independent random variables, a risk averse firm used financial hedging can increase capacity, improve its expected profit, and therefore improve upon its mean-variance objective, by implementing an allocation option. The risk-averse capacity is reduced from the risk neutral capacity.*

Proof: Similar to Proposition 1.3. ■

1.3.4 Numerical Analysis and Managerial Insights

Based on the data setting in Example 3.2, we assume that the risk premium is zero and focus on the insight on the managerial aspect. We summarize the results under different hedging policies in the following table and figures.

	No Allocation Option	Allocation Option	$(A,\cdot)-(-A,\cdot)/(A,\cdot)(\%)$
No Hedge	$U=31.323, X=58.203$ $E=53.818, \lambda V=22.496$	$U=42.277, X=65.039$ $E=68.957, \lambda V=26.679$	$\Delta U=34.97, \Delta X=11.74$ $\Delta E=28.13, \Delta\lambda V=18.60$
Financial Hedge	$U=64.457, X=84.961$ $E=65.758, \lambda V=1.301$	$U=77.710, X=87.695$ $E=79.209, \lambda V=1.499$	$\Delta U=20.56, \Delta X=3.22$ $\Delta E=20.46, \Delta\lambda V=15.21$
$\frac{(\cdot,H)-(\cdot,-H)}{(\cdot,-H)}$ $(\%)$	$\Delta U=105.78, \Delta X=45.97$ $\Delta E=22.19, \Delta\lambda V=-94.2$	$\Delta U=83.81, \Delta X=34.83$ $\Delta E=14.87, \Delta\lambda V=-94.4$	$\frac{(A,H)-(-A,-H)}{(A,H)}(\%)$ $\Delta U=148.10, \Delta X=50.67$ $\Delta E=47.18, \Delta\lambda V=-93.34$

Table 1.2. a) The Effect of Allocation Option and Financial Hedging

As shown in Table 1.2. a), using the allocation option and/or the financial hedging, the capacity, expected profit and mean-variance objective are increased. Comparing the solutions with both hedging and those without any hedging, the increments are significant. The capacity increases more than 50%. The expected profit increases close to 50% and the objective increases close to 150%.

τ		$\frac{(\pm A,-H)}{(-A,-H)}$	$\frac{(\pm A,H)}{(-A,H)}$	$\frac{(-A,\pm H)}{(-A,-H)}$	$\frac{(A,\pm H)}{(A,-H)}$	$\frac{(\pm A,\pm H)}{(-A,-H)}$
2	X^*	0.6000	0.1267	1.1824	0.5368	1.4588
	U^*	1.8645	0.8942	2.3400	1.2087	5.3268
	E^*	1.6179	0.8915	0.7669	0.2766	2.3420
	λV^*	1.3529	0.7605	−0.9246	−0.9435	−0.8672
1.5	X^*	0.1174	0.0322	0.4597	0.3483	0.5067
	U^*	0.3497	0.2056	1.0578	0.8381	1.4810
	E^*	0.2813	0.2046	0.2219	0.1487	0.4718
	λV^*	0.1860	0.1521	−0.9422	−0.9438	−0.9334
1.0	X^*	0.0213	0.0085	0.2580	0.2422	0.2686
	U^*	0.0707	0.0453	0.6452	0.6061	0.7197
	E^*	0.0542	0.0451	0.0928	0.0834	0.1421
	λV^*	0.0233	0.0363	−0.9404	−0.9397	−0.9383
0.5	X^*	0.0023	0.0000	0.1761	0.1733	0.1761
	U^*	0.0060	0.0040	0.4528	0.4499	0.4586
	E^*	0.0048	0.0039	0.0490	0.0480	0.0531
	λV^*	0.0020	0.0000	−0.9331	−0.9332	−0.9331
0.0	X^*	0	0	0.1299	0.1299	0.1299
	U^*	0	0	0.3439	0.3439	0.3439
	E^*	0	0	0.0292	0.0292	0.0292
	λV^*	0	0	−0.9242	−0.9242	−0.9242

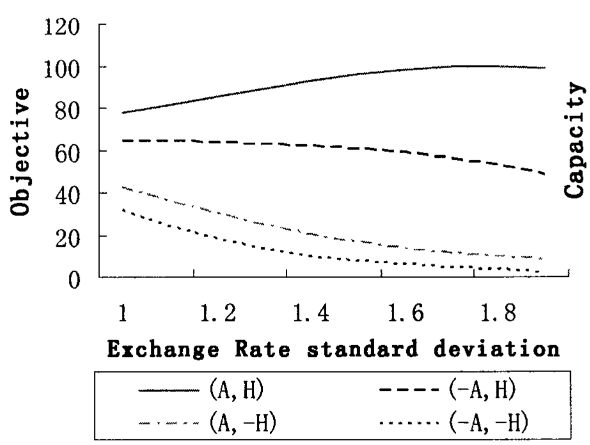
Table 1.2. b) The Comparison of Allocation Option and Financial Hedging

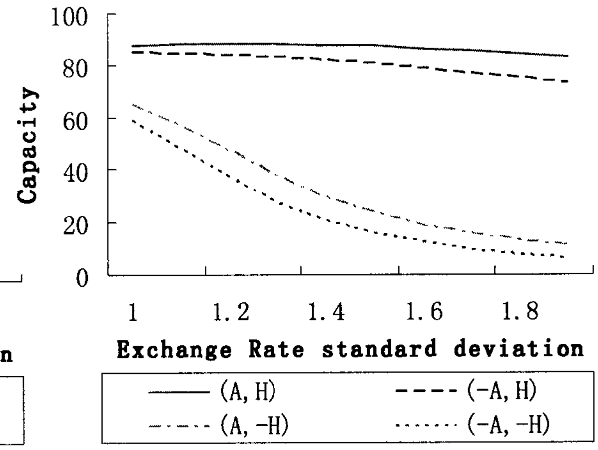
As shown in Table 1.2. b), the effect of allocation option and financial hedging decreases in the allocation cost. Using allocation option can improve the production plan expected profit efficiently if the per unit profit from the foreign market drops down to negative frequently. The financial hedging improves the performance through reducing profit variance. We will illustrate later in Figure 1.2 (a)__(d) that, both of allocation option and financial hedging are insensitive to the demand fluctuation.

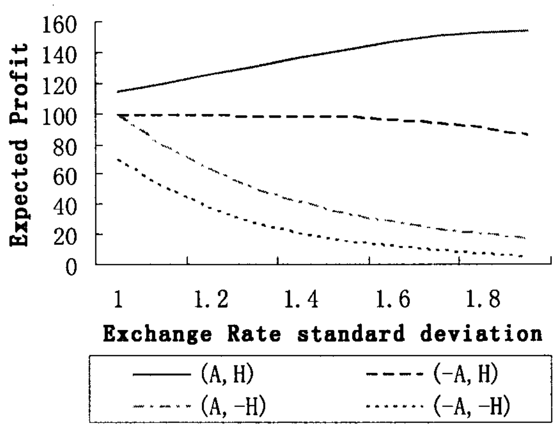
In Figure 1.1 (a)__(d), we increase standard deviation of exchange rate, i.e. $\sigma(\ln s)$, from 1 to 2, while keeping $E(s)=e^3$ and adjusting $\mu(\ln s)$ accordingly. In Figure 1.1 (a), either the allocation option or financial hedging increases the optimal objective value. For the case with both hedging policies, the optimal objective value increases slightly in exchange rate variance, since the allocation option can save more value as the exchange rate variance increases. For the case only with financial hedging, the optimal objective value decreases slightly, since the effect from the remaindered variance is minor. On the other hand, for the cases without financial hedging, the optimal objectives decrease rapidly by the effect of exchange rate variance, no matter with or without the operational hedging. In Figure 1.1 (b), the financial hedging increases the capacity significantly, no matter with or without operational hedging. Capacities with financial hedging are much more stable than those without financial hedging, though all of they decrease in exchange rate variance. In Figure 1.1 (c), the impact on the expected profits from exchange rate is similar to those on the objective illustrated in Figure 1.1 (a). In Figure 1.4 (d), for the cases without financial hedging, the profit variances may decrease with the exchange rate variance since the capacities drop rapidly. For the cases with financial hedging, the profit variances increase slightly in exchange rate variance, since the financial hedging is efficient and capacity is stable.

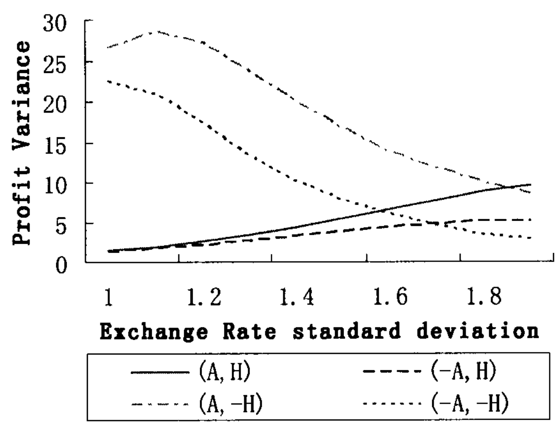
In Figure 1.2 (a) __ (d), we vary the demand standard deviation. All mean-variance objectives, expected profits and capacities decrease with the demand variance. For the case without financial hedging, the profit variance may decrease with the demand variance for the sake of the capacity.

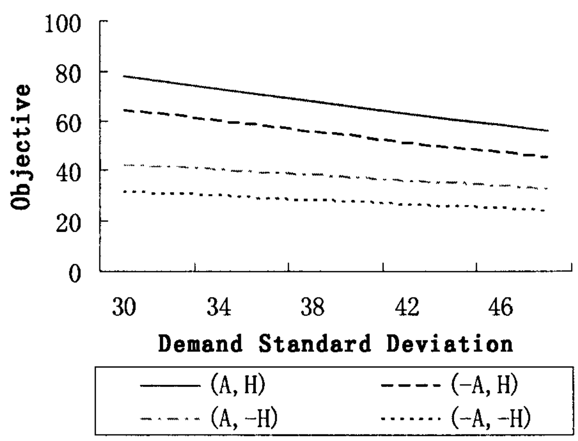
In Figure 1.3 (a) __ (d), the value of λ mainly affects the objective without financial hedge. A reasonable mean-variance ratio should correspond to a reasonable point on the mean-variance frontier. When $\lambda=0$, we obtain the risk neutral solution is $X_N^*=89.6484$, $U_N^*=E_N^*=79.2811$ and $X_{N,-A}^*=86.9141$, $U_{N,-A}^*=E_{N,-A}^*=65.8138$. Figure 1.3 (a) __ (d) clearly illustrate that the solutions of using financial hedging are close to the risk neutral solutions.

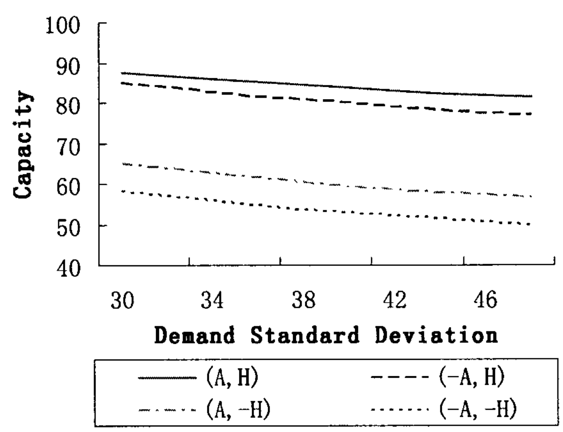

Objective
120
100
80
60
40
20
0
1
1.2
1.4
1.6
1.8
Exchange Rate standard deviation
(A, H)
(-A, H)
(A, -H)
(-A, -H)

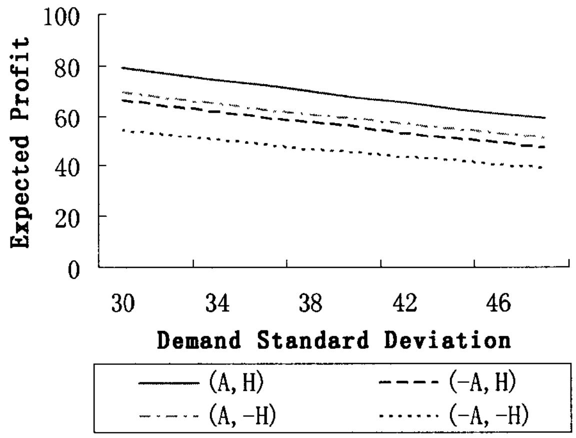

Capacity
100
80
60
40
20
0
1
1.2
1.4
1.6
1.8
Exchange Rate standard deviation
(A, H)
(-A, H)
(A, -H)
(-A, -H)

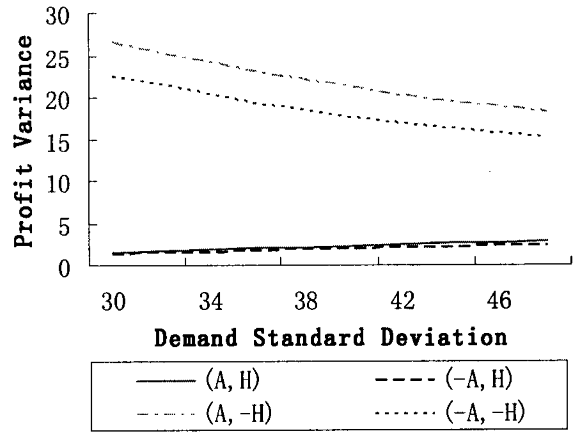

Expected Profit
160
140
120
100
80
60
40
20
0
1
1.2
1.4
1.6
1.8
Exchange Rate standard deviation
(A, H)
(-A, H)
(A, -H)
(-A, -H)

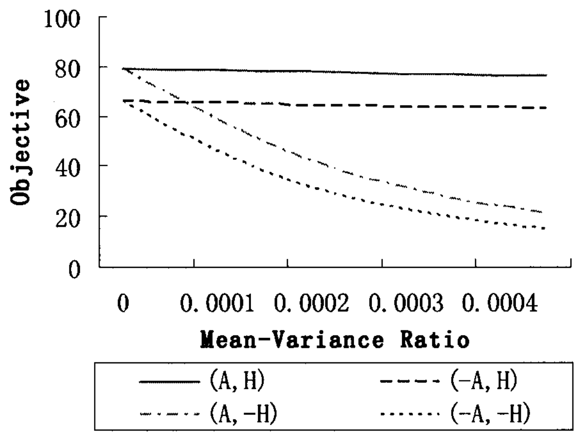

Profit Variance
30
25
20
15
10
5
0
1
1.2
1.4
1.6
1.8
Exchange Rate standard deviation
(A, H)
(-A, H)
(A, -H)
(-A, -H)

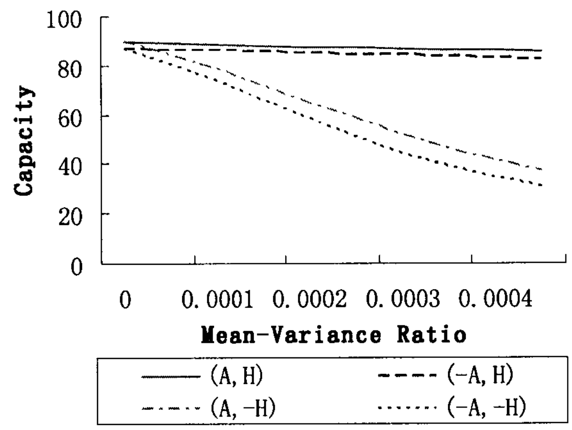

Objective
100
80
60
40
20
0
30
34
38
42
46
Demand Standard Deviation
(A, H)
(-A, H)
(A, -H)
(-A, -H)

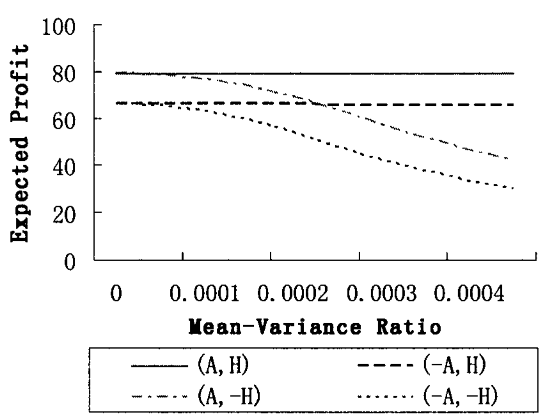

Capacity
100
90
80
70
60
50
40
30
34
38
42
46
Demand Standard Deviation
(A, H)
(-A, H)
(A, -H)
(-A, -H)











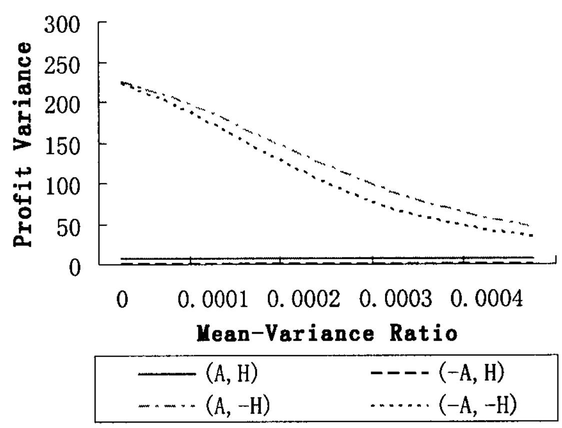


Figure 1.1 (a)__(d) Objective, Capacity, Expected Profit and Profit Variance vs. Exchange Rate (Lognormal) Standard Deviation

Figure 1.2 (a)__(d) Objective, Capacity, Expected Profit and Profit Variance vs. Demand Standard Deviation

Figure 1.3 (a)__(d) Objective, Capacity, Expected Profit and Profit Variance vs. Mean-Variance Ratio

1.4 Hedging the Currency Risk of a Stochastic Demand Domestic Market and a Stochastic Demand Foreign Market

1.4.1 Problem Solution

Similar to Propositions for one stochastic demand foreign market, we present the following propositions.

Proposition 1.5: *For a stochastic demand domestic market and a stochastic demand foreign market, given a capacity* X *and using a group of call option currency contracts at a prespecified exercise price vector* $\hat{S}$*, the optimal hedge size vector* $\hat{Q}(X,\hat{S})$ *is the unique solution to the linear equations*

$$Cov\left[\left(s-\hat{S}\right)^{+},\left(s-\hat{S}\right)^{+}\right]\hat{Q}^{\wedge}=-Cov\left[\pi^{op}\left(X,s,d_1,d_2\right),\left(s-\hat{S}\right)^{+}\right]-\left[\Delta\hat{C}(\hat{S})\right]^{\wedge}/2\lambda \quad (1.14)$$

Proof: Similar to Proposition 1.1. ■

For a given demand distribution, it may be not difficult to choose reasonable exercise price vector to obtain optimal or efficient financial hedging policy.

Example 1.4.1. Based on Example 1.3.1, add the assumption $d_1=d$. If $b=0$, there is a zero-variance hedge policy consisting of two call options

$$(S_1,Q_1)=(\tau_2/p_2,-p_2 min(X-d,a))$$

and $$(S_2,Q_2)=((r_1+\tau_2)/p_2,-p_2[min(X,a)-min(X-d,a)]).$$

Moreover, the hedging policy consisting of $(\tau_2/p_2,Q_1)$ and $((r_1+\tau_2)/p_2,Q_2)$ reduces most profit variance if b is small, where Q_1 and Q_2 are the solution from (1.14).

If b is large, we introduce a complex optimal hedge policy. Given a group of call option exercise price vector as $S=(\tau_2/p_2,\tau_2/p_2+\Delta,\cdots,\tau_2/p_2+n\Delta)$, where Δ is a given small increment, we choose $Q_i = Y_2(S_{i-1}) - Y_2(S_i)$, where $Y_2(s)$ is the number of units allocated to market 2 when the observed exchange rate is s. It is easy to verify that this hedge policy is close to a zero-variance hedge if $\Delta \to 0$ and $\Delta n \to \infty$. ■

We begin to allocate the units to the foreign market only if $s \geq \tau_2 / p_2$ at stage 2. The prior market becomes foreign and there are more units allocated to the foreign market if $s \geq (p_1 - \tau_1 + \tau_2)/p_2$. Thus, we introduce a hedging policy consisting of two call options, which is a natural choice for many practical problems.

Corollary of Proposition 1.5. *For the risk averse firm with a domestic production center and stochastic demands for both markets, given a capacity X and using two call option currency contracts at the prespecified exercise prices $S_1 = \tau_2 / p_2$ and $S_2 = (p_1 - \tau_1 + \tau_2)/p_2$, the optimal hedge sizes are uniquely determined as*

$$Q_1(X, S_1, S_2) = -\frac{\left[\begin{array}{l} Cov\left[\pi^{op}, (s-S_1)^+\right] \cdot V\left[(s-S_2)^+\right] \\ -Cov\left[\pi^{op}, (s-S_2)^+\right] \cdot Cov\left[(s-S_1)^+, (s-S_2)^+\right] \\ +(1/2\lambda)\left(\Delta C(S_1) V\left[(s-S_2)^+\right] - \Delta C(S_2) Cov\left[(s-S_1)^+, (s-S_2)^+\right]\right) \end{array}\right]}{V\left[(s-S_1)^+\right] \cdot V\left[(s-S_2)^+\right] - \left(Cov\left[(s-S_1)^+, (s-S_2)^+\right]\right)^2} \quad (1.15\text{-}a)$$

and

$$Q_2(X, S_1, S_2) = -\frac{\left[\begin{array}{l} V\left[(s-S_1)^+\right] \cdot Cov\left[\pi^{op}, (s-S_2)^+\right] \\ -Cov\left[\pi^{op}, (s-S_1)^+\right] \cdot Cov\left[(s-S_1)^+, (s-S_2)^+\right] \\ +(1/2\lambda)\left(\Delta C(S_1) V\left[(s-S_2)^+\right] - \Delta C(S_2) Cov\left[(s-S_1)^+, (s-S_2)^+\right]\right) \end{array}\right]}{V\left[(s-S_1)^+\right] \cdot V\left[(s-S_2)^+\right] - \left(Cov\left[(s-S_1)^+, (s-S_2)^+\right]\right)^2} \quad (1.15\text{-}b)$$

For the case of demand independent with exchange rate, the optimal hedging sizes are

$$Q_1(X,S_1,S_2) = -p_2 E\left(min\left[(X-d_1)^+, d_2\right]\right)$$

$$-\frac{\Delta C(S_1)V\left[(s-S_2)^+\right] - \Delta C(S_2)Cov\left[(s-S_1)^+,(s-S_2)^+\right]}{2\lambda\left(V\left[(s-S_1)^+\right]\cdot V\left[(s-S_2)^+\right]-\left(Cov\left[(s-S_1)^+,(s-S_2)^+\right]\right)^2\right)} \quad (1.15\text{-}c)$$

and

$$Q_2(X,S_1,S_2) = -p_2 E\left(min[X,d_2] - min\left[(X-d_1)^+, d_2\right]\right)$$

$$-\frac{\Delta C(S_2)V\left[(s-S_1)^+\right] - \Delta C(S_1)Cov\left[(s-S_1)^+,(s-S_2)^+\right]}{2\lambda\left(V\left[(s-S_1)^+\right]\cdot V\left[(s-S_2)^+\right]-\left(Cov\left[(s-S_1)^+,(s-S_2)^+\right]\right)^2\right)} \quad (1.15\text{-}d)$$

Proof: Solve (1.15a-d) by the Cramer rule. ■

1.4.2 Problem Solution with Zero Risk Premium

The strong results for the problem with demand independent to exchange rate are presented in the following remarks.

Remark 1.4. *For the risk averse firm with stochastic demands in the domestic and foreign markets, if the risk premium is zero and the demand is independent with exchange rate, then there exists an optimal hedge for a given capacity* X *among any combination of call options, put options and forward contracts, consisting of the following two call option currency contracts*

$$h_1^* = (Q_1^*, S_1^*):\ Q_1^*(X) = -p_2 E\left(min\left[(X-d_1)^+, d_2\right]\right), S_1^* = \tau_2/p_2, \text{ and} \quad (1.16\text{-}a)$$

$$h_2^* = (Q_2^*, S_2^*):\ Q_2^*(X) = -p_2 E\left(min[X,d_2] - min\left[(X-d_1)^+, d_2\right]\right), \quad (1.16\text{-}b)$$

$$S_2^* = (p_1 - \tau_1 + \tau_2)/p_2$$

Proof: Divide the domain of exchange rate into three sub-ranges, $C_1 \in [0, S_1^*)$, $C_2 \in [S_1^*, S_2^*)$ and $C_3 \in [S_2^*, \infty)$. For the problem conditional on $s \in C_l$, let $R_{h_l}^*(s)$ denote the optimal hedging policy, we have

$$V(\pi^{op} + R_h) \geq \sum_{l=1}^{3} \left[P(s \in C_l) \cdot V_l(\pi^{op} + R_{h_l}^*) \right]$$

where $V_l(\cdot)$ are the expectation value and variance function conditional on $s \in C_l$.

For the problem conditional on $s \in C_l$, the revenue from production market at stage 2 can be rewritten as a linear function of s, $\pi_l^{op} = q_{l1}(X, d_1, d_2) + q_{l2}(X, d_1, d_2)s$. Since the allocation policy is independent with $s \in C_l$, the hedging problem condition on each C_l reduces to similar to the one only with foreign market. Similar to the approach used in Remark 1.1, the hedge policy minimizing $V_l[\pi_l^{op} + R_{h_l}]$ is

$$h_l^* = (Q_l^*, S_l^*): \; Q_l^* = -E_l[q_{l2}] \text{ and } S_l^* = \inf(C_l),$$

where $\inf(\cdot)$ is the infimum function.

It is easy to verify that the hedging policy (h_1^*, h_2^*) given in Remark 1.4 has the same hedging effect as each h_l^* conditional on $s \in C_l$ and it is easy to verify that $E_l[\pi^{op} + R_h^*] = E[\pi^{op} + R_h^*]$. Thus, we have

$$V(\pi^{op} + R_h^*) = \sum_{l=1}^{3} \left[P(s \in C_l) \cdot V_l(\pi^{op} + R_h^*) \right] = \sum_{l=1}^{3} \left[P(s \in C_l) \cdot V_l(\pi_l^{op} + R_{h_l}^*) \right]$$

and the policy (h_1^*, h_2^*) is optimal. ■

Remark 1.5. *For the risk averse firm with stochastic domestic and foreign demands, and zero risk premium, and the demand independent with exchange rate, we have*

a). Using allocation option, the expected profit function is concave.

b). Using allocation option and financial hedging, the profit variance increases in capacity X and $X_{A.H}^* \leq X_N^*$ if the following condition holds

$$\frac{\partial V[\pi^{op}]}{\partial X} = 2Cov(o, \pi^{op}) \geq 0 \qquad (1.17)$$

$$\text{where } o(X,s,d_1,d_2) \equiv \begin{cases} r_2(s) & (s,d_1,d_2) \in D_2(X) \\ r_1 & (s,d_1,d_2) \in D_1(X), \\ 0 & (s,d_1,d_2) \in D_0(X) \end{cases}$$

and $D_2(X) = \{(r_2 \geq r_1, X < d_2) \cup (r_1 > r_2 \geq 0, d_1 \leq X < d_1 + d_2)\}$,

and $D_1(X) = \{(r_1 > r_2, X < d_1) \cup (r_2 \geq r_1, d_2 \leq X < d_1 + d_2)\}$,

and $D_0(X) = \{(r_2 \geq 0, X \geq d_1 + d_2) \cup (r_2 < 0, X \geq d_1)\}$,

whose are the domains of exchange rate and demand realization when the marginal unit allocates to the foreign market, domestic market and no market, respectively.

Proof. a). The operational hedging allocates the marginal unit to the available location with highest marginal profit. Thus, the marginal expected profit decreases with capacity and the concavity holds.

b). Observe from the proof in Remark 1.4

$$\frac{\partial V\left[\pi^{op}(X,s,d_1,d_2) + R_{h^*}(s)\right]}{\partial X} = \sum_{l=1}^{3}\left\{P(s \in C_l)\frac{\partial V_l\left[\pi_l^{op}(X,s,d_1,d_2) + R_{h_l^*}(s)\right]}{\partial X}\right\}.$$

Define $A_l(X) = q_{l1} + q_{l2}s - E[q_{l2}](s - S_l^*)$, we have

$$\begin{aligned}
\frac{\partial V_l}{2\partial X} &= E_l\left(A_l\frac{\partial A_l}{\partial X}\right) - E_l[A_l]\frac{\partial E_l[A_l]}{\partial X} \\
&= E_l\left[(q_{l1} + q_{l2}s)\frac{\partial A_l}{\partial X}\right] - E_l\left[E[q_{l2}]s\frac{\partial A_l}{\partial X}\right] + S_l^* E[q_{l2}]\frac{\partial E_l[A_l]}{\partial X} \\
&\quad - E_l[q_{l1} + q_{l2}s]\frac{\partial E_l[A_l]}{\partial X} + E_l[q_{l2}]E_l(s)\frac{\partial E_l[A_l]}{\partial X} - S_l^* E[q_{l2}]\frac{\partial E_l[A_l]}{\partial X} \\
&= E_l\left[(q_{l1} + q_{l2}s)\frac{\partial A_l}{\partial X}\right] - E_l\left[E[q_{l2}]s\frac{\partial A_l}{\partial X}\right] - E_l[q_{l1}]\frac{\partial E_l[A_l]}{\partial X} \\
&= E_l\left(\left[\pi_l^{op} - E(\pi_l^{op})\right]\frac{\partial A_l}{\partial X}\right)
\end{aligned}$$

In each sub-area C_l, let $(D_{l1}(X), D_{l2}(X), D_{l0}(X))$ denote the corresponding domains of demand, in which the marginal unit is allocated to the foreign, domestic and no market, respectively. Using the optimal financial policy $Q_l^* = -E_l[q_{l2}]$ and $S_l^* = \inf(C_l)$ given in Remark 1.4. We have

$$\frac{\partial A_l(X)}{\partial X} = o_l - p_2 P(D_{l2})(s - S_l^*)$$

where $o_l = \begin{cases} r_2 & (d_1,d_2) \in D_{l2}(X) \\ r_1 & (d_1,d_2) \in D_{l1}(X) \\ 0 & (d_1,d_2) \in D_{l0}(X) \end{cases}$ is the marginal allocation function for C_l.

and $$\frac{\partial V_l}{2\partial X} = E_l \begin{pmatrix} o_l\left[\pi_l^{op} - E(\pi_l^{op})\right] \\ -p_2 P(D_{l2})(s - S_l^*)\left(\left[q_{l1} - E_l(q_{l1})\right] + \left[q_{l2} - E_l(q_{l2})\right]s\right) \end{pmatrix}$$

$$= E_l\left(o_l\left[\pi_l^{op} - E(\pi_l^{op})\right]\right) = Cov(o_l, \pi_l^{op})$$

and $$\frac{\partial V\left[\pi^{op}\right]}{\partial X} = \sum_{l=1}^{3}\left[P(s \in C_l) 2Cov(o_l, \pi_l^{op})\right] = 2Cov(o, \pi^{op})$$

Thus, the result in b) follows if the condition in (1.17) holds. ■

In many practical environments of using the allocation option and financial hedging, the marginal profit variance increases with an accelerating speed while the marginal expected profit is positive. Then, the optimal solution can be easily obtained from the first order condition.

Example 1.4.2: Based on the data setting in Example 1.3.2, we add the domestic market: $d_1 \sim N(100,30)$ and independent with d_2 and s. Choose a larger $\lambda = 0.002$ to emphasis the impact of profit variance and the results are shown in Figure 1.4. ■

However, there exist some special counter examples even for the independent case.

Example 1.4.3. Assume that $ce^{\gamma T} = 3$, $\tau_1 = \tau_2 = 0$, $p_1 = 10$, $p_2 = 1$, $\ln(s) \sim N(1.5, 0.5)$ and the risk premium is zero and $\lambda = 0.005$. The demand is a two stage distribution dependent on a location factor δ: $(d_1,d_2)\sim(100\delta,200(1-\delta))$ and $P(\delta=0)=P(\delta=1)=0.5$.

As shown in Figure 1.5, there exist multiple local solutions by the compensation effect between the two markets. The condition in (1.17) fails and the monotone of profit variance is broken. Note that the marginal variance increases again when the capacity is very large by the risk effect from the exchange rate uncertainty. ■

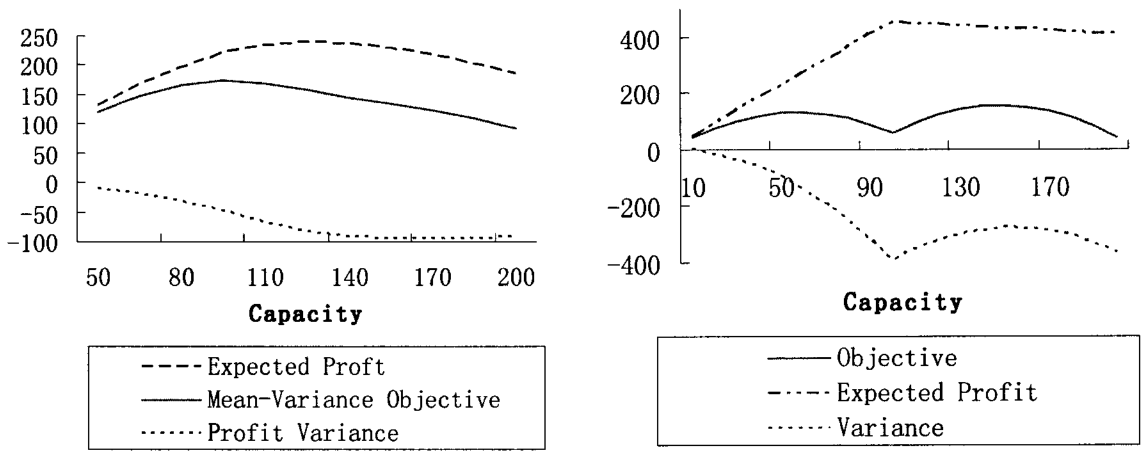


Figure 1.4__1.5 Objective, Expected Profit and Profit Variance vs. Capacity

On the other hand, the optimal capacity for the risk neutral problem is provided in the following remark.

Remark 1.6. *For the risk neutral firm with stochastic domestic and foreign demands and zero risk premium, the optimal risk-neutral capacity is unique if the density function is positive on the whole domain.*

If the demand and exchange rate being independent random variables, X_N^* *solves*

$$\begin{bmatrix} E[\max(r_1,r_2)]\cdot P(d_1,d_2\in D_{N1})+r_1\cdot P(d_1,d_2\in D_{N2}) \\ +E\left[r_2^+\right]\cdot P(d_1,d_2\in D_{N3})+E\left[\min\left(r_1,r_2^+\right)\right]\cdot P(d_1,d_2\in D_{N4}) \end{bmatrix}=ce^{\gamma T} \qquad (1.18\text{-}a)$$

where $D_{N1}=\{X<d_1, X<d_2\}$, $D_{N2}=\{X<d_1, X\ge d_2\}$, $D_{N3}=\{X\ge d_1, X<d_2\}$ *and* $D_{N4}=\{X\ge d_1, X\ge d_2, X<d_1+d_2\}$.

For the deterministic demand problem, the risk neutral capacity is

$$X_N^*=\begin{cases} 0 & \text{if } E\left[\max(r_1,r_2)\right]\le ce^{\gamma T} \\ d_1 & \begin{cases}\text{if } E\left[\max(r_1,r_2)\right]>ce^{\gamma T}, E\left[r_2^+\right]\le ce^{\gamma T}, d_1<d_2, \text{ or} \\ \text{if } r_1>ce^{\gamma T}, E\left[\min\left(r_1,r_2^+\right)\right]\le ce^{\gamma T}, d_1\ge d_2\end{cases} \\ d_2 & \begin{cases}\text{if } E\left[\max(r_1,r_2)\right]>ce^{\gamma T}, r_1\le ce^{\gamma T}, d_2<d_1, \text{ or} \\ \text{if } E\left[r_2^+\right]>ce^{\gamma T}, E\left[\min\left(r_1,r_2^+\right)\right]\le ce^{\gamma T}, d_2\ge d_1\end{cases} \\ d_1+d_2 & \text{if } E\left[\min\left(r_1,r_2^+\right)\right]>ce^{\gamma T}\end{cases} \qquad (1.18\text{-}b)$$

Proof.

$$\begin{aligned}\partial E(\pi)/\partial X &= -ce^{-\gamma T} \\ &+ \iint_{D_{N1}\cup[0,\infty)} \max(r_1,r_2) f(s,d_1,d_2) ds dd + \iint_{D_{N2}\cup[0,\infty)} r_1 f(s,d_1,d_2) ds dd \\ &+ \iint_{D_{N3}\cup[0,\infty)} r_2^+ f(s,d_1,d_2) ds dd + \iint_{D_{N4}\cup[0,\infty)} \min\left(r_1,r_2^+\right) f(s,d_1,d_2) ds dd\end{aligned}$$

Since the area with higher profit rate shrinks as X increasing, $\partial E(\pi)/\partial X$ decreases and X_N^* is unique if $f(s,d)>0$ on the whole domain. The results for the cases of independent demand and fixed demand follow directly through some algebra. ■

For the risk averse problem with deterministic demands and zero risk premium, the financial hedge policy reduces the profit variance to zero and the unique risk neutral solution is optimal.

1.4.3 Role of the Production and Financial Hedges

1.4.3.1 Impact of Production Allocation

The allocation option increases the expected profit as well as the mean-variance objective in most of practical cases. However, it does not always increase the optimal capacity.

We can illustrate the effect of allocation option on the capacity by considering the risk neutral problem. Given a pair of deterministic demands $(\bar{d}_1, \bar{d}_2)$, the production decision is separated into two single market problems if no using allocation option. Similar to (1.9-c) in Remark 1.2, the optimal capacity for each market is determined by the marginal expected profit at the single market as

$$X^*_{1,N,-A} = \begin{cases} \bar{d}_1 & r_1 \geq ce^{\gamma T} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad X^*_{2,N,-A} = \begin{cases} \bar{d}_2 & E(r_2) \geq ce^{\gamma T} \\ 0 & \text{otherwise} \end{cases} \tag{1.19-a}$$

Let $X_{N,-A} = X_{1,N,-A} + X_{2,N,-A}$ denote the total capacity. For the case with $E\left[\min\left(r_1, r_2^+\right)\right] < ce^{\gamma T} \leq \min\left[r_1, E\left(r_2\right)\right]$, comparing with (1.18-b) and (1.19-a), we have

$$X^*_N = \max\left[\bar{d}_1, \bar{d}_2\right] \leq \bar{d}_1 + \bar{d}_2 = X^*_{N,-A}.$$

On the other hand, for the case with $E\left[\max\left(r_1, r_2\right)\right] \geq ce^{\gamma T} > \max\left[r_1, E\left(r_2\right)\right]$, we have

$$X^*_N = \min\left[d_1, d_2\right] > 0 = X^*_{N,-A}$$

In the other cases, the allocation does not change the optimal risk neutral capacity.

The optimal risk neutral objective of no using allocation option is

$$E\left[\pi^{op}_{N,-A}(X^*_{N,-A}, s, \bar{d}_1, \bar{d}_2)\right] = \left(r_1 - ce^{\gamma T}\right)^+ \bar{d}_1 + \left[E\left(r_2\right) - ce^{\gamma T}\right]^+ \bar{d}_2 \tag{1.19-b}$$

The optimal risk neutral objective of using allocation option is

$$\begin{aligned} E\left[\pi^{op}_N(X^*_N, s, \bar{d}_1, \bar{d}_2)\right] &= \left(E\left[\max\left(r_1, r_2\right)\right] - ce^{\gamma T}\right)^+ \min\left(\bar{d}_1, \bar{d}_2\right) \\ &+ \left(r_1 - ce^{\gamma T}\right)^+ \left(\bar{d}_1 - \bar{d}_2\right)^+ + \left[E\left(r_2^+\right) - ce^{\gamma T}\right]^+ \left(\bar{d}_2 - \bar{d}_1\right)^+ \\ &+ \left(E\left[\min\left(r_1, r_2^+\right)\right] - ce^{\gamma T}\right)^+ \left[\left(\bar{d}_1 + \bar{d}_2\right) - \max\left(\bar{d}_1, \bar{d}_2\right)\right] \end{aligned} \tag{1.19-c}$$

The incremental objective value is

$$\Delta_{N,\pm A} = E\left[\pi_N^{op}(X_N^*,s,\bar{d}_1,\bar{d}_2)\right] - E\left[\pi_{N,-A}^{op}(X_{N,-A}^*,s,\bar{d}_1,\bar{d}_2)\right]$$
$$= \begin{bmatrix} \left(E\left[\max(r_1,r_2)\right]-ce^{\gamma T}\right)^+ + \left(E\left[\min(r_1,r_2^+)\right]-ce^{\gamma T}\right)^+ \\ -\left(r_1-ce^{\gamma T}\right)^+ - \left[E(r_2)-ce^{\gamma T}\right]^+ \end{bmatrix} \min\left(\bar{d}_1,\bar{d}_2\right) \qquad (1.19\text{-d})$$
$$+\left(\left[E(r_2^+)-ce^{\gamma T}\right]^+ - \left[E(r_2)-ce^{\gamma T}\right]^+\right)\left(\bar{d}_2-\bar{d}_1\right)^+$$

It is easy to verify that $\Delta_{N,\pm A} \geq 0$ by considering each case of whether $\left(E\left[\max(r_1,r_2)\right]-ce^{\gamma T}\right)^+$, $\left(r_1-ce^{\gamma T}\right)^+$, $\left[E(r_2)-ce^{\gamma T}\right]^+$ and $\left(E\left[\min(r_1,r_2^+)\right]-ce^{\gamma T}\right)^+$ are positive.

Since the revenue is increased by using allocation option for all observation at stage 2, the allocation hedging always increases the expected profit for the stochastic demand problems.

These arguments can be adopted for the risk-averse problems either with financial hedging or without financial hedging when the mean-variance ratio is small.

The allocation option increases capacity and decreases profit variance in many practical problems. However, different from the one foreign market problem, the capacity may be brought down and the profit variance may also be increased by the allocation option in some special environments.

Example 1.4.4. Assume that $ce^{\gamma T}=1$, $\tau_1=\tau_2=0$, $p_1=e^{1.5}$, $p_2=1$ and the risk premium is zero. Given a joint uniform demand distribution: $d_2\in[0,100]$ and $d_1=100-d_2$. Assuming $\ln(s)\sim N(\mu,\sigma)$ and $\lambda=0.02$, we increase $\sigma\in(0.15,0.6)$ and adjust μ accordingly to keep $E(s)=e^{1.5}$. Then, we compare the capacities in Figure 1.6 (a). The risk averse capacity no using allocation option is higher than the risk neutral solution, since there is a natural operational hedging effect by the demand compensation when the capacities in both markets are increasing. The allocation option can satisfy the market compensation and bring down the capacity in this example. After financial hedging, the capacity of using allocation option is more close to the risk neutral solution. In Figure 1.6 (b), the allocation option may increases the variance for the cases no using financial hedging, since it improves and the marginal expected profit and increases the expected number of units allocated to the foreign market. The allocation option may also reduce the capacity and increase the variance for the cases using financial hedging as illustrated in Figure (1.6) (c) and (d). ■

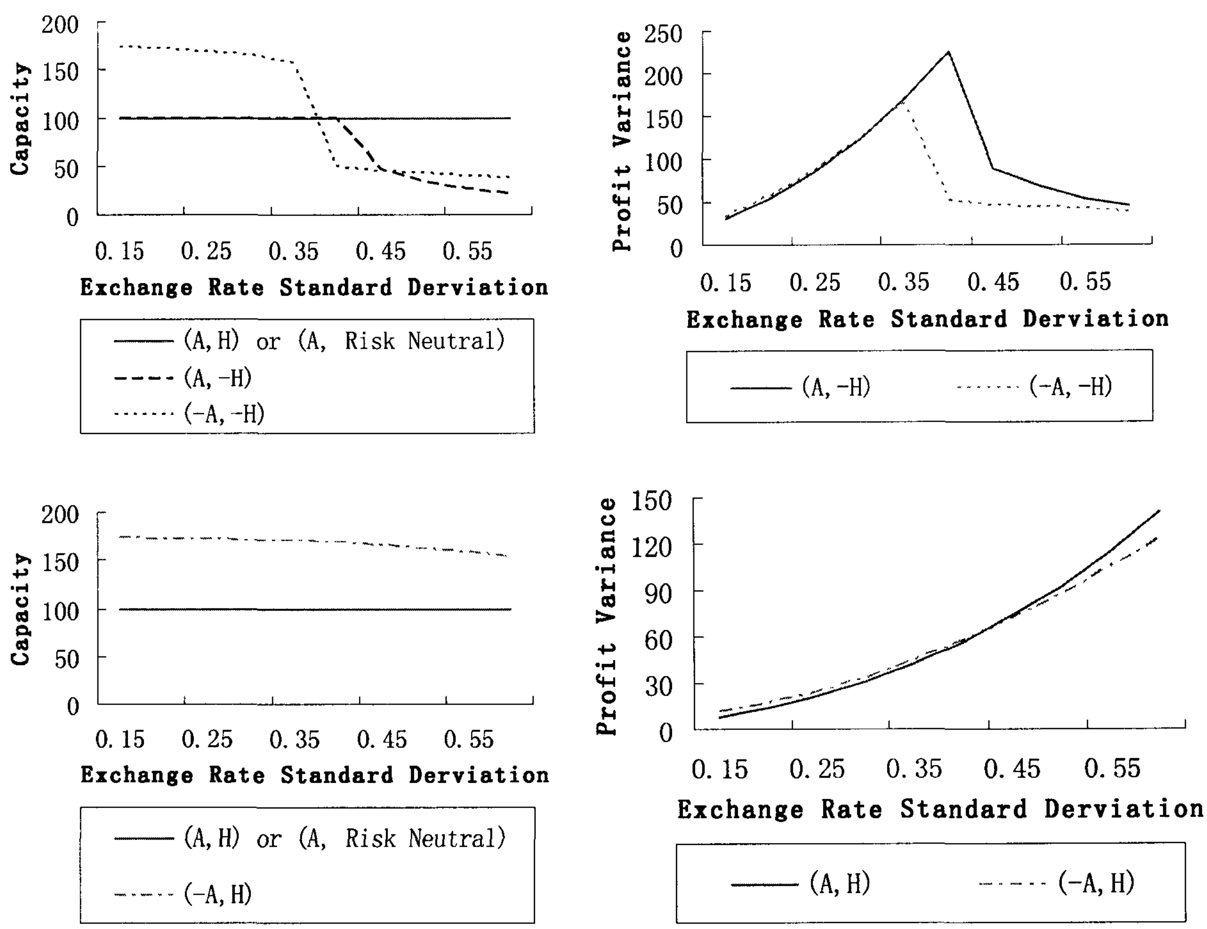


Figure 1.6 Allocation Option Effect in Two Market Problem

a) and d) Capacity; b) and c) Profit Variance vs. Exchange Rate Standard Derivation

1.4.3.2 Impact of Financial Hedging

The use of a financial hedge achieves its obvious intent to reduce the profit variance given capacity, and consequently increase the mean-variance objective value. For the case with zero risk premium, the financial hedging does not affect the expected profit, we focus on the hedging effect on the variance. The following strong results report the impact on the capacities for the independent cases either using or no using an allocation option.

Remark 1.7. *For the risk averse firm with stochastic domestic and foreign demands, and zero risk premium, and the demand independent with exchange rate, and using of allocation, the use of (optimal) financial hedge increases the capacity.*

Proof: Observe from the proof in Remark 1.5

$$\frac{\partial V\left[\pi^{op}\right]}{2\partial X}-\frac{\partial V\left[\pi^{op}+R_{h^*}\right]}{2\partial X}=\sum_{l=1}^{3}\left\{P(s\in C_l)\cdot\left(\frac{\partial V_l\left[\pi_l^{op}\right]}{2\partial X}-\frac{\partial V_l\left[\pi_l^{op}+R_{h_l^*}\right]}{2\partial X}\right)\right\}$$

$$=\sum_{l=1}^{3}\left\{P(s\in C_l)\begin{bmatrix}-E_l\left(R_{h_l^*}\frac{\partial\left(\pi_l^{op}+R_{h_l^*}\right)}{\partial X}+\pi_l^{op}\frac{\partial R_{h_l^*}}{\partial X}\right)\\+E_l\left(R_{h_l^*}\right)\frac{\partial E_l\left(\pi_l^{op}+R_{h_l^*}\right)}{\partial X}+E_l\left(\pi_l^{op}\right)\frac{\partial E_l\left(R_{h_l^*}\right)}{\partial X}\end{bmatrix}\right\}$$

Observing $\pi_l^{op}=q_{l1}+q_{l2}s$, $R_{h_l^*}=-E[q_{l2}](s-S_l^*)$, $\frac{\partial\pi_l^{op}}{\partial X}=o_l$, $\frac{\partial R_{h_l^*}}{\partial X}=-p_2P(D_{l2})(s-S_l^*)$,

we have the argument similar to Proposition 1.4

$$\begin{aligned}\frac{\partial V\left[\pi^{op}\right]}{2\partial X}-\frac{\partial V\left[\pi^{op}+R_{h^*}\right]}{2\partial X}&=-E_l\begin{pmatrix}\left(-E_l[q_{l2}](s-S_l^*)\right)\left(o_l-p_2P(D_{l2})(s-S_l^*)\right)\\+(q_{l1}+q_{l2}s)\left(-p_2P(D_{l2})(s-S_l^*)\right)\end{pmatrix}\\&\quad+E_l\left(-E_l[q_{l2}](s-S_l^*)\right)E_l\left(o_l-p_2P(D_{l2})(s-S_l^*)\right)\\&\quad-E_l(q_{l1}+q_{l2}s)\,p_2P(D_{l2})E_l(s-S_l^*)\\&=E_l[q_{l2}]\left[E_l\left(\begin{Bmatrix}r_1(s-S_l^*) & D_{l1}\\ r_2(s-S_l^*) & D_{l2}\\ 0 & D_{l0}\end{Bmatrix}\right)-E_l\left(\begin{Bmatrix}r_1E_l(s-S_l^*) & D_{l1}\\ r_2E_l(s-S_l^*) & D_{l2}\\ 0 & D_{l0}\end{Bmatrix}\right)\right]\\&\quad+p_2P(D_{l2})E_l\left[(q_{l1}+q_{l2}s)(s-S_l^*)\right]-E_l(q_{l1}+q_{l2}s)\,p_2P(D_{l2})E_l(s-S_l^*)\\&\quad-p_2P(D_{l2})E_l[q_{l2}]E\left[(s-S_l^*)^2\right]+p_2P(D_{l2})E_l[q_{l2}]\left[E(s-S_l^*)\right]^2\\&=p_2E_l[q_{l2}]V_l(s)\geq 0\end{aligned}$$

■

Remark 1.8. *For the risk averse firm with stochastic domestic and foreign demands, and zero risk premium, and the demand independent with exchange rate, and no using of allocation, then,*

a) the forward contract with size $Q_{-A}^{f*}(X) = -p_2 E\left(\min\left[X_{2,-A}, d_2\right]\right)$ *is the optimal financial hedging for a given capacity vector* $\left(X_{1,-A}, X_{2,-A}\right)$ *among any combination of call options, put options and forward contract;*

b) the use of financial hedge increases the capacity at foreign market, i.e. $X_{2,-A}$.

Proof: a). Rewrite the revenue from production market at stage 2

$$\pi_{-A}^{op}(X_{1,-A}, X_{2,-A}, s, d_1, d_2) = r_1 \min\left(X_{1,-A}, d_1\right) + r_2 \min\left(X_{2,-A}, d_2\right)$$

as a linear function of s,

$$\pi_{-A}^{op} = q_{1,-A}(X_{1,-A}, X_{2,-A}, d_1, d_2) + q_{2,-A}(X_{1,-A}, X_{2,-A}, d_1, d_2)s.$$

Since the allocation policy is independent with s, the hedging problem is simplified similar to the one foreign market case. Applying the approach used in Remark 1.1 and Remark 1.4, the hedge policy minimizing $V\left[\pi_{-A}^{op} + R_{h,-A}\right]$ is a forward contract

$$h_{-A}^{*} = h_{-A}^{f*}\left(Q_{-A}^{f*}\right):\ Q_{-A}^{f*}(X) = -E\left(q_{2,-A}\right) = -p_2 E\left[\min\left(X_{2,-A}, d_2\right)\right].$$

b). Similar to the proof in Remark 1.5

$$\frac{\partial V\left[\pi_{-A}^{op}\right]}{2\partial X_{i,-A}} - \frac{\partial V\left[\pi_{-A}^{op} + R_{h_{-A}^{f*}}\right]}{2\partial X_{i,-A}} = \begin{bmatrix} -E\left(R_{h_{-A}^{f*}} \dfrac{\partial\left(\pi_{-A}^{op} + R_{h_{-A}^{f*}}\right)}{\partial X_{i,-A}} + \pi_{-A}^{op} \dfrac{\partial R_{h_{-A}^{f*}}}{\partial X_{i,-A}}\right) \\ +E\left(R_{h_{-A}^{f*}}\right) \dfrac{\partial E_l\left(\pi_{-A}^{op} + R_{h_{-A}^{f*}}\right)}{\partial X_{i,-A}} + E_l\left(\pi_{-A}^{op}\right) \dfrac{\partial E_l\left(R_{h_{-A}^{f*}}\right)}{\partial X_{i,-A}} \end{bmatrix} \geq 0.$$

Observing $\pi_{-A}^{op} = q_{1,-A} + q_{2,-A}s$, $R_{h_{-A}^{f*}} = -E\left[q_{2,-A}\right]\left[s - E(s)\right]$, $\dfrac{\partial \pi_{-A}^{op}}{\partial X_{i,-A}} = \begin{cases} r_i & X_{i,-A} < d_i \\ 0 & X_{i,-A} \geq d_i \end{cases}$,

$\dfrac{\partial R_{h_{-A}^{f*}}}{\partial X_{1,-A}} = 0$ and $\dfrac{\partial R_{h_{-A}^{f*}}}{\partial X_{2,-A}} = -p_2 P\left(X_{2,-A} < d_2\right)\left[s - E(s)\right]$, we have the argument similar to Remark 1.7

$$\frac{\partial V\left[\pi_{-A}^{op}\right]}{2\partial X_{1,-A}}-\frac{\partial V\left[\pi_{-A}^{op}+R_{h_{-A}^{f*}}\right]}{2\partial X_{1,-A}}$$

$$=-E\left(-E\left[q_{2,-A}\right]\left[s-E(s)\right]\left(\begin{cases} r_1 & X_{1,-A}<d_1 \\ 0 & X_{1,-A}\geq d_1 \end{cases}\right)\right)$$

$$+E\left(-E\left[q_{2,-A}\right]\left[s-E(s)\right]\right)E\left(\begin{cases} r_1 & X_{1,-A}<d_1 \\ 0 & X_{1,-A}\geq d_1 \end{cases}\right)$$

$$=0$$

$$\frac{\partial V\left[\pi_{-A}^{op}\right]}{2\partial X_{2,-A}}-\frac{\partial V\left[\pi_{-A}^{op}+R_{h_{-A}^{f*}}\right]}{2\partial X_{2,-A}}$$

$$=-E\left(\begin{array}{l} -E\left[q_{2,-A}\right]\left[s-E(s)\right]\left(\begin{cases} r_2 & X_{2,-A}<d_2 \\ 0 & X_{2,-A}\geq d_2 \end{cases}-p_2P\left(X_{2,-A}<d_2\right)\left[s-E(s)\right]\right) \\ +\left(q_{1,-A}+q_{2,-A}s\right)\left(-p_2P\left(X_{1,-A}<d_2\right)\left[s-E(s)\right]\right) \end{array}\right)$$

$$+E\left(-E\left[q_{2,-A}\right]\left[s-E(s)\right]\right)E\left(\begin{cases} r_2 & X_{2,-A}<d_2 \\ 0 & X_{2,-A}\geq d_2 \end{cases}-p_2P\left(X_{1,-A}<d_2\right)\left[s-E(s)\right]\right)$$

$$-E\left(q_{1,-A}+q_{2,-A}s\right)p_2P\left(X_{2,-A}<d_2\right)E\left[s-E(s)\right]$$

$$=E\left[q_{2,-A}\right]E\left[\left(\begin{cases} r_2\left(s-E(s)\right) & X_{2,-A}<d_2 \\ 0 & X_{2,-A}\geq d_2 \end{cases}\right)-E\left(s-E(s)\right)E\left(\begin{cases} r_2 & X_{2,-A}<d_2 \\ 0 & X_{2,-A}\geq d_2 \end{cases}\right)\right]$$

$$-p_2P\left(X_{2,-A}<d_2\right)E\left[q_{2,-A}\right]E\left[\left(s-E(s)\right)^2\right]$$

$$+p_2P\left(X_{2,-A}<d_2\right)E\left(\left(q_{1,-A}+q_{2,-A}s\right)\left[s-E(s)\right]\right)$$

$$+p_2P\left(X_{2,-A}<d_2\right)E\left[q_{2,-A}\right]E\left[s-E(s)\right]E\left[s-E(s)\right]$$

$$-p_2P\left(X_{2,-A}<d_2\right)E\left[\left(q_{1,-A}+q_{2,-A}s\right)\right]E\left[s-E(s)\right]$$

$$=p_2P\left(X_{2,-A}<d_2\right)E\left[q_{2,-A}\right]V(s)$$

$$\geq 0$$

Given a pair of capacity $\left(X_{1,-A}^{'},X_{2,-A}^{'}\right)$ with $X_2^{'}\leq X_{2,-A-H}^{*}$, we have

$$
\begin{aligned}
& U_{-A}\left(X_{1,-A,-H}^{*}, X_{2,-A,-H}^{*}\right)-U_{-A}\left(X_{1,-A}^{'}, X_{2,-A}^{'}\right) \\
& \geq\left[U_{-A}\left(X_{1,-A,-H}^{*}, X_{2,-A,-H}^{*}\right)-U_{-A,-H}\left(X_{1,-A,-H}^{*}, X_{2,-A,-H}^{*}\right)\right] \\
& \quad-\left[U_{-A}\left(X_{1,-A}^{'}, X_{2,-A}^{'}\right)-U_{-A,-H}\left(X_{1,-A}^{'}, X_{2,-A}^{'}\right)\right] \\
& \geq\left[V_{-A,-H}\left(X_{1,-A,-H}^{*}, X_{2,-A,-H}^{*}\right)-V_{-A}\left(X_{1,-A,-H}^{*}, X_{2,-A,-H}^{*}\right)\right] \\
& \quad-\left[V_{-A,-H}\left(X_{1,-A}^{'}, X_{2,-A}^{'}\right)-V_{-A}\left(X_{1,-A}^{'}, X_{2,-A}^{'}\right)\right] \\
& =\int_{X_{2,-A}^{'}}^{X_{2,-A,-H}^{*}}\left(\frac{\partial V_{-A,-H}\left(X_{1,-A,-H}^{*}, X_{2,-A}\right)}{2 \partial X_{2,-A}}-\frac{\partial V_{-A}\left(X_{1,-A,-H}^{*}, X_{2,-A}\right)}{2 \partial X_{2,-A}}\right) \cdot d X_{2,-A} \\
& \quad+\int_{X_{1,-A}^{'}}^{X_{1,-A,-H}^{*}}\left(\frac{\partial V_{-A,-H}\left(X_{1,-A}, X_{2,-A}^{'}\right)}{2 \partial X_{2,-A}}-\frac{\partial V_{-A}\left(X_{1,-A}, X_{2,-A}^{'}\right)}{2 \partial X_{2,-A}}\right) \cdot d X_{1,-A} \\
& \geq 0
\end{aligned}
$$

This implies that $\left(X_{1,-A}^{'}, X_{2,-A}^{'}\right)$ is not the solution and result in b) holds. ■

For the problem of no using allocation option, we illustrate some interesting managerial insights through the following example.

Example 1.4.5. Based on the data setting in Example 1.4.4, we increase $\sigma \in (0.15, 0.6)$ compare the capacities related to financial hedging in Figure 1.7 (a). The risk averse capacity either using or no using financial hedging may be higher than the risk neutral capacity for the sake of the demand compensation. Furthermore, the financial hedging pushes the capacities far away from the risk neutral solution. There exist multiple local optimal plans for no hedging case and potentially for the hedging case as shown in Figure 1.7 (c). (given $\sigma = 0.3$). ■

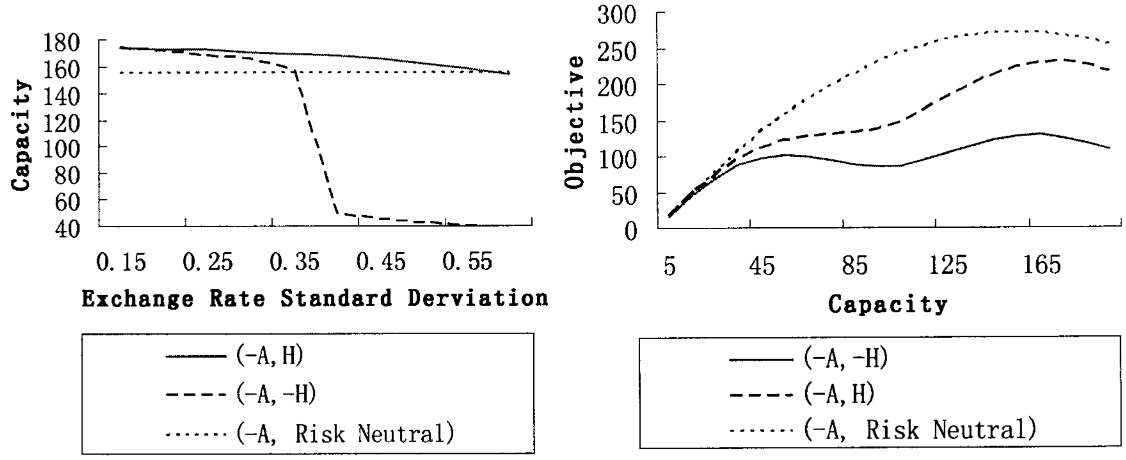


Figure 1.7 (a). Capacity vs. Exchange Rate Standard Derivation.

(b). Objective vs. Capacity

1.4.3.3 Comparison of Production Allocation and Financial Hedging

Based on the data setting in Example 1.4.2, we summarize the results under different hedging policies in the following tables.

τ		$\frac{(\pm A,-H)}{(-A,-H)}$	$\frac{(\pm A,H)}{(-A,H)}$	$\frac{(-A,\pm H)}{(-A,-H)}$	$\frac{(A,\pm H)}{(A,-H)}$	$\frac{(\pm A,\pm H)}{(-A,-H)}$
2	X^*	−0.1119	−0.2620	0.3315	0.1065	−0.0173
	U^*	1.2707	1.2732	0.2542	0.2556	1.8510
	E^*	1.6199	1.3286	0.1462	0.0188	1.6690
	λV^*	5.0009	8.0126	−0.8993	−0.8488	−0.0927
1.5	X^*	−0.2272	−0.2823	0.1861	0.1016	−0.1487
	U^*	0.8767	0.7552	0.3423	0.2554	1.3561
	E^*	0.9481	0.7984	0.0989	0.0144	0.9762
	λV^*	1.2532	5.0531	−0.9418	−0.8437	−0.6477
1.0	X^*	−0.2662	−0.2497	0.1203	0.1456	−0.1594
	U^*	0.5487	0.4638	0.3246	0.2519	0.9389
	E^*	0.5628	0.5030	0.0547	0.0144	0.5853
	λV^*	0.6150	4.2107	−0.9479	−0.8320	−0.7286
0.5	X^*	−0.2092	−0.2716	0.0863	0.0005	−0.2088
	U^*	0.3347	0.2950	0.2856	0.2473	0.6648
	E^*	0.3705	0.3282	0.0319	0.0000	0.3706
	λV^*	0.5089	3.5206	−0.9483	−0.8450	−0.7662
0.0	X^*	−0.1900	−0.1825	0.0668	0.0765	−0.1280
	U^*	0.2445	0.2226	0.2492	0.2272	0.5273
	E^*	0.2751	0.2547	0.0201	0.0037	0.2798
	λV^*	0.4045	3.4493	−0.9471	−0.8325	−0.7647

Table 1.3. a). Varying Allocation Cost and Comparing Hedging Effect.

$\sigma(d_i)$		$\frac{(\pm A,-H)}{(-A,-H)}$	$\frac{(\pm A,H)}{(-A,H)}$	$\frac{(-A,\pm H)}{(-A,-H)}$	$\frac{(A,\pm H)}{(A,-H)}$	$\frac{(\pm A,\pm H)}{(-A,-H)}$
10	X^*	–0.3657	–0.4291	0.1473	0.0326	–0.3450
	U^*	0.6397	0.4371	0.4224	0.2467	1.0442
	E^*	0.6158	0.4572	0.1124	0.0032	0.6210
	λV^*	0.5305	22.6835	–0.9954	–0.9289	–0.8912
20	X^*	–0.3181	–0.3531	0.1623	0.1027	–0.2481
	U^*	0.7379	0.5813	0.3764	0.2523	1.1764
	E^*	0.7401	0.6091	0.0969	0.0143	0.7649
	λV^*	0.7487	7.4854	–0.9786	–0.8961	–0.8183
30	X^*	–0.2272	–0.2823	0.1861	0.1016	–0.1487
	U^*	0.8767	0.7552	0.3423	0.2554	1.3561
	E^*	0.9481	0.7984	0.0989	0.0144	0.9762
	λV^*	1.2532	5.0531	–0.9418	–0.8437	–0.6477
40	X^*	–0.1585	–0.2319	0.2069	0.1016	–0.0730
	U^*	1.0788	0.9810	0.3214	0.2593	1.6177
	E^*	1.2165	1.0444	0.1046	0.0188	1.2582
	λV^*	1.8486	4.5737	–0.8910	–0.7868	–0.3926
50	X^*	–0.1058	–0.1576	0.2274	0.1562	–0.0339
	U^*	1.3857	1.3455	0.3125	0.2904	2.0785
	E^*	1.5954	1.4415	0.1140	0.0480	1.7200
	λV^*	2.6056	5.2996	–0.8426	–0.7249	–0.0082

Table 1.3. b). Varying Demand Standard Deviation and Comparing Hedging Effect.

As shown in Table 1.3. a), the effect of allocation option decreases in the allocation cost. The allocation option can improve expected profit through reduce the total capacity. The variance control depends on both the production allocation and the marginal expected profit. So the performance is complex.

As shown in Table 1.3. b), the effect of allocation option increases in the demand variance. The variance control is stable as the demand variance increases.

1.5 Foreign Production Center Problem

1.5.1 Model and Solution

Now we consider the case where the production facility is in the foreign country (market 2). We use the subscripts $-D$ is a reminder that the variables in this expression are related to the foreign production center. Assume the incremental profit per unit sold at the second stage in market 1 (in home currency) and market 2 (in foreign currency) are

$$r_{1,-D}(s) = p_1 - s\tau_{1,-D} \text{ and } r_{2,-D}(s) = s(p_2 - \tau_{2,-D})$$

The results for the foreign production center can be obtained similar to the argument for the counterpart problems with domestic production center in Section 1.4.

1.5.2 Impact of Allocation Option

The optimal location of production center may shift from foreign to domestic or from domestic to foreign by the effect of allocation option even for the risk neutral problem with fixed demands, which is illustrated in the following two examples.

Example 1.5.1. Consider a risk neutral problem with a foreign market problem such that $d=1$, $p=1$, $ce^{\gamma T}=1$, $\tau=2$, $c_{-D}e^{\gamma T}=2.9$, $\tau_{-D}=0$, $E[r]>0$ and $E[r_{-D}]>0$. It is easy to verify that the risk neutral optimal capacities for all cases either with or without allocation option at both locations is $X_N^* = X_{N,-A}^* = X_{N,-D}^* = X_{N,-A,-D}^* = 1$.

For the case without allocation option, the separation of the profits between the domestic and foreign production centers is

$$\Delta_{N,-A,\pm D} = E\left(r - ce^{\gamma T}\right) - E\left(r_{-D} - c_{-D}e^{\gamma T}\right) = c_{-D}e^{\gamma T} - \tau - ce^{\gamma T} = -0.1 < 0$$

and the optimal location of production center is foreign.

Comparing with the cases with allocation option, the allocation policy is the same if the location is foreign. If the location is domestic, the units are sent to market 2 only if

$s \geq \tau$ and the expected total profit increases $\int_0^{\tau}(\tau - s)e(s)\,ds > 0$. The optimal location of production center shifts from foreign to domestic, if $\int_0^{\tau}(\tau - s)e(s)\,ds > 0.1$. ■

Example 1.5.2. Assume that $ce^{\gamma T} = 2.35$, $\tau_1 = 0$, $\tau_2 = \infty$, $c_{-D}e^{\gamma T} = 1$, $\tau_{1,-D} = 1.25$, $\tau_{2,-D} = 0$, $p_1 = 3.4$, $p_2 = 1$ and $d_1 = d_2 = 1$. Assume $\ln(s) \sim N(\mu, 1)$ satisfying $E(s) = 2$. It is easy to see that $p_1 \geq ce^{\gamma T} + \tau_1$, $E(sp_2) < ce^{\gamma T} + \tau_2$, $p_1 < c_{-D}e^{\gamma T} + E(s\tau_{1,-D})$ and $E(sp_2 - \tau_{2,-D}) \geq c_{-D}e^{\gamma T}$. For the case without allocation option, the optimal capacities are $X^*_{N,-A} = d_1 = 1$ and $X^*_{N,-A,-D} = d_2 = 1$ for the domestic and foreign locations from the first order condition, and results

$$\Delta_{N,-A,\pm D} = \left(p_1 - ce^{\gamma T}\right)d_1 - \left(E\left[s\left(p_2 - \tau_{2,-D}\right)\right] - c_{-D}e^{\gamma T}\right)d_2 = 0.05 > 0.$$

The optimal location of production center is domestic.

For the case with allocation option, if the production center is domestic, the optimal capacity and expected profit are still $X^*_N = 1$ and $(p_1 - c)d_1$ from $\tau_2 = \infty$. If the location is foreign, given $X_{N,-D} = 1$, the units are sent to market 1 if $s \leq p_1/(1 + \tau_{1,-D}) = 1.5$. The expected total profit increases $2.25\int_0^{1.5}[1.5 - s]e(s)\,ds > 0.05$. Thus, the optimal location of production center shifts from domestic to foreign. ■

Intuitively, the impact of allocation option on one location may be more significant than the other one, which results the shift of production center.

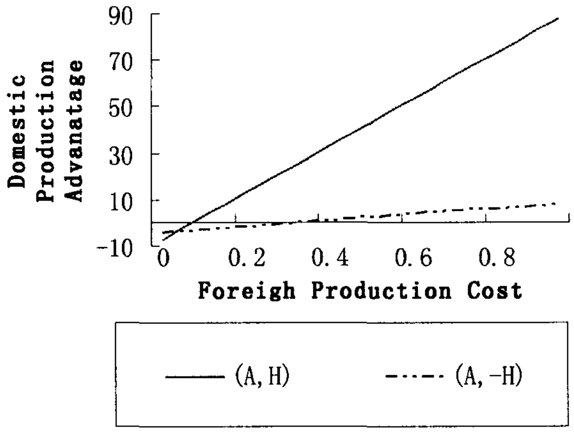
1.5.3 Impact of Financial Hedging

By the same intuition, the optimal location of production center may shift from foreign to domestic or from domestic to foreign by the effect of the financial hedging. Intuitively, the risk related to exchange rate is larger when the production center located in foreign country. Thus, the financial hedging may shift the production center from local to foreign by the effect of financial hedging. However, for some special cost structure, the financial hedging have a more significant effect when the production location is local. We illustrate this counter situation in the following two examples.

Example 1.5.3 considers the case of using allocation option and Example 1.5.4 considers the case of no using allocation option.

Example 1.5.3. Assume that $ce^{\gamma T}=1$, $p_1=3$, $p_2=1$, $\tau_1=0$, $\tau_2=1.2$ $d_1=d_2=100$, $\tau_{1,-D}=\infty$, $\tau_{2,-D}=0$ and $\ln(s)\sim N(1,1.2)$. Let $\Delta U_{-H,\pm D}=U^*_{-H}-U^*_{-H,-D}$ and $\Delta U_{-H,\pm D}=U^*_{-H}-U^*_{-H,-D}$ denote the domestic production advantage. The optimal production location is domestic if and only if $\Delta U_{-H,\pm D}\geq 0$ or $\Delta U_{H,\pm D}\geq 0$. We report the production advantages for $c_{-D}\in(0,1]$ in Figure 1.8 (a), the optimal location of production center is shifted from foreign to domestic by the financial hedging effect as $c_{-D}e^{\gamma T}\in(0.1,0.3]$. Intuitively, the local production center supplies both markets and bear larger risk. By using financial hedging, the objective can be increased not only from variance control, but also the increment of capacity. Thus, the production center may shift from foreign to domestic. ■

Example 1.5.4. Assume that $ce^{\gamma T}=2$, $\tau_1=\tau_2=\tau_{2,-D}=0$, $\tau_{1,-D}=\infty$, $p_1=5$, $p_2=1$ and $\ln(s)\sim N(1.5,0.5)$. The demand distribution is $(d_1,d_2)\sim(100\delta,100(1-\delta))$ and $P(\delta=0)=P(\delta=1)=0.5$. We report the domestic production advantages for $c_{-D}\in[0.5,1.5]$ in Figure 1.8 (b). The optimal production center shift from foreign to domestic by the financial hedging effect as $c_{-D}\in(0.95,1)$, since the local production center can increases the capacity and improve the objective more efficiently after the financial hedging controls the risk. ■



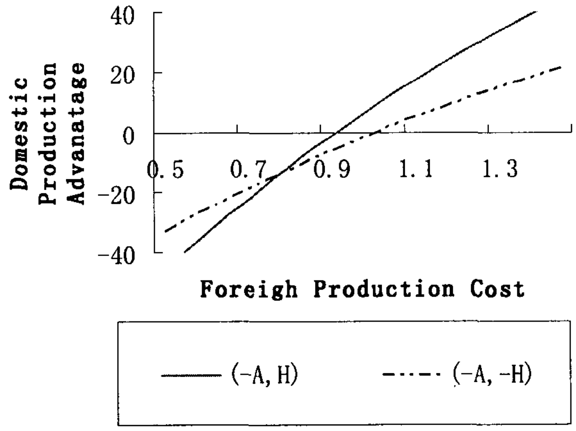


Figure 1.8 (a) __ (b) Domestic Production Advantage vs. Foreign Production Cost.

1.6 Managerial Insights & Conclusions

Within our stylized modeling framework, we have been able to quantify and better understand the effects of a comprehensive risk management approach (via simultaneous operational and financial hedging) to demand and exchange rate uncertainty in supplying a domestic market and a foreign market. Our research clearly demonstrates that, when demand uncertainty enters the picture, the risk averse firm can no longer ignore financial hedging issues in making its capacities. Similarly, financial hedges, based on arbitrary capacity assumptions, such as producing to meet expected demand, fail to provide effective total risk control.

Our research starts from a stochastic foreign market. Given a capacity and using a group of call option currency contracts at the prespecified exercise prices, the optimal hedging sizes are the unique solution to the linear equations. The optimal capacity is uniquely determined by close formulation for the independent case (i.e. demand being independent with exchange rate) when using a single call option with exercise price equal to the ratio of the localization cost and selling price, i.e. τ/p.

Our analysis clearly shows that the financial hedge consists of two separated parts: one minimizing the profit variance and the other balancing the risk premium. The risk premium is small in the most practical environments and its effect on hedging size is independent with production decisions. Thus, we present strong results by ignoring the risk premium and focusing on minimizing the profit variance. For the independent case with zero risk premium, the optimal financial hedging policy, among all combinations of call options, put options and forwards contracts, is a single call option with exercise price equal to τ/p, and the optimal hedge size simplifies to $Q^*(X) = G^1(X) - G^1(0)$, where $G^1(X)$ is the demand loss function. The optimal capacity is uniquely determined by a close formulation.

When demand and exchange rates could be correlated, the optimal hedging is complex. On the other hand, for most of the cases, well designed call option currency contracts, as prescribed by our formulas, prove very effective in controlling total risk.

Our analysis clearly establishes the values of allocation option and financial hedging for risk averse firms. For the independent case, the use by a firm of either an operational hedge (via an allocation option) or a financial hedge will affect its optimal capacities. For the case without financial hedging, the use of an allocation option favors increased production, in an effort to improve expected profits and the objective value. Similarly, the use of a financial hedge, as intuitively expected, allows an increased capacity to improve expected profits and the objective value through better control of profit variance no matter using or no using an allocation option if the risk premium is positive. For the cases using financial hedge, using allocation option increases capacity, and improves expected profit and the objective value if the risk premium is zero.

For a domestic market and a foreign market, given a capacity and using a group of call option currency contracts at the prespecified exercise prices, the optimal hedge size vector is the unique solution to the linear equations. For the independent case with zero risk premium, the optimal hedge for a given capacity X among any combination of call options, put options and forward contracts, consists of the two call option currency contracts at one exercise price equal to τ_2/p_2 and another equal to the per unit profit at the local market. However, there may exist multiple local optimal capacities. The financial hedging increases the capacity for the case in the presence of allocation option, and increases the capacity for the foreign market when no using allocation option. The allocation option may increase or decrease capacity even if the demands are constants.

If the production center is located in foreign country, we can obtain the similar results by the symmetric structure of the considered model with domestic production center. Our examples illustrate that both allocation option and financial hedging can shift the production location from domestic to foreign or from foreign to domestic.

Our research makes an important first step in closing an apparent gap in the international operation and finance literature on quantifying the simultaneous setting of operational and financial hedging policy parameters, and explaining the nature of implications of such practices for capacity decisions of global firms.

The present paper has dealt with the simplified environment of a firm supplying from a production center, a home market and a foreign market. An obvious, but non-trivial, extension of the same research theme is to address how the presence of multiple production centers and multiple foreign market destinations affect the nature of the operational and financial hedging practices. It is always a question of interest to understand if there exists more than one firms in competitive environments, the use, or the lack of use, of operational and/or financial hedges in the presence of demand and exchange rate uncertainty could have serious repercussions not only on the magnitude of production output but also on fundamental facility network structural choices, such as the desired location of production facilities. We have started exploring some of these issues in our ongoing research efforts.

1.7 References

Bigger, N., J. Hull. 1983. The valuation of currency options. *Financial Management.* Spring issue 24-27.

Black, F., M. Scholes. 1973. The pricing of options and corporate liabilities. *J. Political Econom.* **81** 637-659.

Carter, J. R., S. K. Vickery. 1988. Managing volatile exchange rates in international purchasing. *J. of Purchasing and Materials Management.* Winter 13-20.

Carter, J. R., S. K. Vickery. 1989. Currency exchange rates: their impact on global sourcing. *J. of Purchasing and Materials Management.* Fall 13-20.

Chen, F. and A. Federgruen. 2000. Mean-variance analysis of basic inventory models, Technical report, Graduate School of Business, Columbia University.

Chowdhry, B. and J. T. B. Howe. 1999. Corporate risk management for multinational corporations: financial and operational hedging policies, *European Finance Review.* **2** 229-246.

Cohen, M. A., A. Huchzermeier. 1999. Global supply chain management: a survey of research and applications. in Tayur, S, et al. (eds.) *Quantitative Models for Supply Chain Management.* 670-702.

Cohen, M. A., S. Mallik. 1997. Global supply chains: research and applications. *Production and Oper. Management.* **6**(2) 193-210.

Dixit, A. K. 1989. Entry and exit decisions under uncertainty. *J. Political Econom.* **97**(3) 620-638.

Dixit, A. K., R. S. Pindyck. 1994. *Investment under uncertainty.* Princeton University Press, Princeton N. S.

Dornier, P., R. Ernst, M. Fender, P. Kouvelis. 1998. *Global Operations and Logistics: Text and Cases.* Wiley, NY.

Flood, E. Jr., D. R. Lessard. 1986. On the measurement of operational exposure to exchange rates: a conceptual approach. *Financial Management.* Spring 25-36.

Grauer, F. L., R. H. Litzenberger, R. E. Stehle. 1976. Sharing rules and equilibrium in an international capital market under uncertainty. *J. Financial Econom.* **3** 233-256.

Hanoch, G., H. Levy. 1969. The efficiency analysis of choices involving risk. *Rev. Econom. Stud.* **36** 335-346.

Harrison, J. M., J. A. Van Mieghem. 1999. Multi-resource investment strategies: operational hedging under demand uncertainty. *Euro. J. of Oper. Res.* **113**(1) 17-29.

Hertzell, S., C. Casper. 1988. Rethinking financial Strategies: coping with unpredictable exchange rates. *The Mckinsey Quart.* Summer 12-24.

Hodder, J. E. 1982. Exposure to exchange rate movements. *J. of Internat. Econom.* **13** 375-386.

Hodder, J. E. 1984. Financial market approaches to facility location under uncertainty. *Oper. Res.* **32** 1374-1380.

Hodder, J. E., M. C. Dincer. 1986. A multifactor model for international plant location and financing under uncertainty. *Comput. Oper. Res.* **13**(5) 601-609.

Huchzermeier, A., M. A. Cohen. 1996. Valuing operational flexibility under exchange rate uncertainty. *Oper. Res.* **44**(1) 100-113.

Jensen, M. C., W. H. Meckling. 1976. Theory of the firm: managerial behavior, agency costs and ownership structure. *J. Financial Econom.* **3** 305-360.

Jucker, J. V., R. C. Carlson. 1976. The simple plant-location problem under uncertainty. *Oper. Res.* **24** 1045-1055.

Kogut, B., N. Kalatilaka. 1994. Operating flexibility, global manufacturing and the option value of a multinational network. *Management Sci.* **40**(1) 123-139.

Kouvelis, P. 1999. Global sourcing strategies under exchange rate uncertainty. in Tayur, S. et al. (eds.) *Quantitative Models for Supply Chain Management.* 625-668.

Kouvelis, P., K. Axarloglou, V. Sinha. 2001. Exchange rates and ownership strategies. *Management Sci.* **47**(8) 1063-1080.

Lessard, D. R., J. B. Ligltstone. 1986. Volatile exchange rates can put operations at risk. *Harvard Bus. Rev.* July-August 107-114.

Li, C-L, P. Kouvelis. 1999. Flexible and risk-sharing supply contracts. *Management Sci.* **45**(10) 1378-1398.

Markowitz, H. M. 1952. Portfolio selection. *J. Finance.* **7** 77-91.

Markus, A. J. 1982. Risk sharing and theory of the firm. *Bell J. Econom.* **13** 369-378.

Mello, A. S., J. E. Parsons and A. J. Triantis. 1995. An integrated model of multinational flexibility and financial hedging. *Journal of International Economics.* **39** 27-51.

Merforn, R. C. 1973. Theory of rational option pricing. *Bell J. of Econom. and Management Sci.* **4** 141-183.

O'Brien, T. J. 1996. *Global Financial Management.* Wiley, New York, NY.

Philippatos, G. C., N. Gressis. 1975. Conditions of equivalence among E-V, SSD and E-H portfolio selection criteria: the case for uniform, normal and lognormal Distributions. *Management Sci.* **21** 617-625.

Porter, R. B., J. E. Gaumnitz. 1972. Stochastic dominance vs. mean variance portfolio selection analysis: An empirical evaluation. *Amer. Econom. Rev.* **62** 438-445.

Scheller-Wolf, A., S. Tayur. 1997. Reducing international risk through quantity contracts. Working Paper, GSIA, Carnegie Mellon University, Pittsburgh, PA.

Shapiro, A. 1988. *International Financial Management.* Allyn and Bacon, MA.

Silver, E. A., D. F. Pyke, R. Peterson. 1998. *Inventory management and Capacityning and scheduling, Third Edition.* Wiley, NY.

Solnik, B. 1991. *International Investments, 2nd edition.* Addison-Wesley Publishing Co, Reading, MA.

Sundaram, A. K., J. S. Black. 1995. *The International Business Environment: Text and Cases.* Preactice Hall.

Triantis, J. Alexander, J. E. Hodder. 1990. Valuing flexibility as a complex option. *J. Finance.* **45** (June) 549-565.

Tsay, A. A., S. Nahmias, N. Agrawal. 1999. Modeling supply chain contracts: a review. in Tayur, S. et al. (eds.) *Quantitative Models for Supply Chain Management.* Kluwer, Norwell, MA 299-336.

Van Mieghem, J. A. 1998. Investment strategies for flexible resource. *Management Sci.* **44**(8) 1071-1078.

Van Mieghem, J. A. 1999. Coordinating investment, production and subcontracting. *Management Sci.*

Van Mieghem, J. A. 2003. Operational hedging of capacity portfolio: A mean-variance analysis, Technical report, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Chapter 2

Dynamic Pricing through Customer Discounts for Optimizing Multi-Class Customers Demand Fulfillment

2.1 Introduction

2.1.1 Problem Motivation

In this paper we study the problem of allocating inventory over a period to demand from several classes of customers when partial backlogging of unfilled demand is possible. The customer classes are distinguished by the price they are to pay for the item and their willingness to wait for fulfillment of demand in a subsequent period. Demand from each customer class is modeled as a realization of a (non-stationary) random variable during each of several stages a period is divided into. The firm is able to view this demand in each stage prior to making an allocation decision on which demand to fill. Unfilled demand may then be backlogged for fulfillment. The probability of this occurring is influenced by a discount the firm may offer as well as some class specific parameters.

The problem arises in a number of industries. The motivating example is based on the fulfillment of demand at a wholesaler of industrial products. At the firm's distribution center, orders are received throughout the day from customers for whom there is a fixed price for a unit (generally from contractual terms or competitive environment) and who expect same day shipping. Given a limited inventory, the distributor may choose to offer the customer next day shipping on the item in hope of being able to fulfill the request of a more valued customer. In order to induce the customer to wait for supply, the distributor may offer a discount. Similar problems are found in on-line catalog and direct market channel businesses where firms need to determine their availability to ship on a given day. For example, an on-line bookseller may quote a time until shipping that is based on the ability of the firm to withdraw a unit of demand from a warehouse. Customers arriving early in the day may be quoted a longer time until shipping so that inventory may be reserved for customers coming later in the day with higher valued orders.

In this paper we show how to determine the inventory allocation, dynamic discounts to offer based on inventory availability and class-based demand realization, and customer-class prioritization (referred to as the ADP problem) in a model where each period is divided into a number of stages. We show that inventory should be allocated in each stage in class-order as long as the inventory is above a determined threshold level. These class-based threshold levels are shown to be monotonic in the waiting demand and the inventory remaining. The initial inventory level is determined by a base-stock ordering policy. For the continuous time demand case, we provide an efficient and robust heuristic for its solution in real time. For every arriving customer a simple rule, executed with real time efficiency, can be used to determine if the customer is allocated immediately inventory, and if not what discount to be given in an effort to have him wait (backorder) rather than walk away (lost sale). The paper contributes to the literature by incorporating dynamic price discounting (i.e. offering economic incentives for customer retention fully reflecting all available inventory and realized demand information) with inventory rationing in a model which can be solved in an efficient manner. Our solution approach executed in a near optimal heuristic way can address the real time revenue management needs of agile distribution and direct market channels.

2.1.2 Literature Review

The research is related to work in inventory rationing, offering economic incentives for customer retention, revenue management and dynamic pricing. Early work by Topkis (1968) considered the rationing of inventory to demand from n customer classes when a period is divided into several intervals. He shows that a base-stock ordering policy is optimal and demand is fulfilled in class order as long as inventory is above a class-dependent allocation level. A model similar to Topkis under a different operating environment has been considered by Frank, Zhang and Duenyas (1999). Cohen, Kleindorfer and Lee (1988) consider an (s, S) inventory system where two classes of customers arrive, with the higher priority customer being served first. The focus of the paper is on the determination of the reorder level s and the order-up-to value S through the development of heuristics and approximations. Deshpande, Cohen and Donohue (2003) analyze a static threshold-based rationing policy for a continuous review two demand classes system with backorders. Ha (1997), considers the problem of allocating inventory to n customer classes in a make-to-stock environment where stock replenishment is explicitly modeled as a production system through a $M/E_k/1$ queue. The optimal policy is characterized by an inventory level below which production is initiated and an inventory level for each customer class above which demand will be fulfilled for the class. De Vericourt, Karaesmen, and Dallery (2001) investigate various stock allocation polices in make-to-stock production systems and assess the benefits of inventory rationing polices. Gerchak, Parlar and Yee (1985) consider a two concurrently-arriving class problem where the decision is when to reject lower class customers based on the time to go and inventory. Weatherford, Bodily and Pfeifer (1993) study the determination of dynamic allocation limits in a two-class problem with the possibility of lower-class customers purchasing at a higher class price. In all these papers, the unsatisfied demand is either entirely backlogged or entirely lost. On the other hand, we consider the partial backlogging case where price discounts are used to induce a desirable level of backlogging. In a recent paper Cattani and Souza (2002) investigate inventory-rationing policies in multi-class a-priori determined fixed price environments with application for firms operating in a direct market channel. They compare the performance of these rationing policies with a pure first-come, first-serve policy under various scenarios for customer response to delay (lost sales, backlog, and a combination of lost sales and backlog). The inventory system is fed by a production system or co-located supplier with exponentially distributed processing times. Our paper emphasizes the use of dynamic policies in combination with inventory rationing in partial backlogging environments, with the backlog level affected by the dynamic pricing polices of the firm.

Another related stream of work in the inventory management literature looks on economic incentives to retain customers in the presence of stockouts. Cheung (1998) considers a continuous review model where a discount can be offered to the customers willing to accept backorders even before the inventory is depleted, but the proportion of backordering customers is not a function of monetary incentives, as is the case in our work. The optimality of offering backordering incentives for a simple inventory system is explored in DeCroix and Arreola Risa (1998), but their analysis does not exploit different customer classes or dynamic discount adjustments. Further, their analysis seems to imply that the majority of cost savings is as a result of offered backordered incentives after the stockout occurs, which is not necessarily true in a multi-class profit differentiated customer demand environment as the one examined in our paper.

Chen (2001) inspired by e-retailing environments studies optimal pricing-replenishment strategies that balance the costs due to discounted prices and the benefits due to advance demand information from customers willing to accept longer lead times for the right discount. In the paper the firm offers a menu of price and lead-time combinations, and customers can choose their priorities. Chen focuses on finding an optimal menu of static prices, as opposed to dynamic discounts we use, and he does not consider inventory-rationing policies, which are essential elements of our formulation. Wang, Cohen and Zheng (2002) study a problem of meeting demand from two demand classes with different lead time requirements. However, their paper focuses on studying required inventory levels in a two echelon supply chain, where each location follows a base stock policy with no inventory rationing, whereas we specifically focus on dynamic pricing and inventory rationing for a single location.

The problem is also related to the well-studied revenue management problem in which a firm seeks to determine the number of units of capacity to reserve for sale to customers arriving at a later time. A lot of that work was motivated by the airline industry practices, and sets prices for flights in the presence of multiple customer classes. Belobaba (1987) considered a heuristic approach to solving the multiple fare-class problem. Wollmer (1992), Brumelle and McGill (1993) and Robinson (1995), considered extensions and refinements determining optimal solutions. In all these models fare classes are assumed to arrive sequentially so that the solution of the problem requires determining how much to reserve for other fare-classes. A review of the extensive revenue management literature is provided by McGill and van Ryzin (1999). In our paper, we assume concurrent arrivals of demand from different classes.

A number of recent papers study a dynamic version of the perishable asset revenue management problem where the selling price may be varied continuously over time. Gallego and van Ryzin (1994) consider a model where the demand rate for an item depends on the current price offered and solve for an expected revenue maximizing policy. Bitran and Mondschein (1997) study a similar problem with non-time homogenous demand and demonstrate the effectiveness of restricting the number of prices to a small set and enforcing monotonic policies with respect to the price over time. Zhao and Zheng (2000) study an extension of Gallego and van Ryzin (1994) with nonhomogenous reservation prices. Finally, Feng and Xiao (2000) find an exact solution for the multiple-price model in continuous time when monotonic pricing policies are assumed. Our work is related to this work as we search for allocation policies in a multiple-price model, where we explicitly incorporate the time-dimension. However, our model differs from this work in that we offer dynamically adjusted price discounts only to the customers denied prompt service. Further, while previous papers have assumed rejected customers are lost, we allow for rejected customers to wait, i.e., they are (partially) backlogged with a probability dependent on a price discount.

Outside of the revenue management context, there has been work on integrating dynamic pricing with production/supply policy but mostly for single product homogenous customer populations (e.g., Zabel (1972), Thowsen (1975)). In more recent work, Chan, Simchi-Levi and Swann (2001) consider a multi-period deterministic demand model where pricing and production decisions must be made for each period with some capacity constraint. Similarly, Federgruen and Heching (1999) in a stochastic demand setting focuses on showing optimality of a policy where a base-stock is ordered up-to and a list price is charged.

2.1.3 Paper Organization

The remainder of the paper is as follows. In Section 2.2 we formally introduce the model. We present a solution to the problem in Section 2.3. In Section 2.4 we consider the continuous-time version of the problem. In Section 2.5 we discuss properties of the ADP policies and provide useful insights from some numerical experiments. We draw our final conclusions in Section 2.6.

2.2 Model

Based on the problem description, we consider a model in which there are K customer classes and M stages in each period. Let subscript $i \in I = \{1, \cdots, K\}$ denote the customer class and subscript $j \in \{1, \cdots, M\}$ represent the stage. Let the per unit revenue from class i be p_i and, without loss of generality, assume $p_1 \leq \cdots \leq p_K$. Let d_{ij} be the demand from customer class i in stage j and let $d_j = \{d_{1j}, \cdots, d_{Kj}\}$ be the demand vector in stage j. We assume that each customer orders exactly one unit. We assume that the demand distribution in each stage is known and independent of previous stages. Let X_j denote the supply at the start of stage j and let Y_j be the inventory allocated in stage j to demand.

Let z_{ij} be the discount offered to a class-i customer in stage j if they are not allocated supply promptly and let $z_j = \{z_{1j}, \cdots, z_{Kj}\}$. Thus a customer, initially denied supply, will pay $p_i - z_{ij}$ if he commits to wait for delivery. Let the probability that a customer will wait be γ_{ij} and we assume for simplicity that it is defined by the linear function

$$\gamma_{ij} = \alpha_{ij} + \beta_{ij} z_{ij}$$

That is, those customers not receiving supply choose to wait based on independent Bernouli random variables. We restrict $0 \leq z_{ij} \leq \dfrac{1-\alpha_{ij}}{\beta_{ij}}$ so that $0 \leq \gamma_{ij} \leq 1$. We also assume that α_{ij} and β_{ij} are non-decreasing in i, i.e. where $0 \leq \alpha_{Kj} \leq \cdots \leq \alpha_{1j} \leq 1$ and $0 \leq \beta_{Kj} \leq \cdots \leq \beta_{1j}$. Thus customers with lower prices are more likely to wait. If a customer chooses to wait, the firm may still allocate supply to that customer in a subsequent stage or may not allocate supply until the end of the period.

We adopt the following costs. The per unit cost to the seller is c_p. If the number of units allocated in stage j is larger than a given delivery capacity, g_j, the firm incurs a per extra unit delivered marginal cost, c_j, which we refer to as a congestion cost. Such cost results from unplanned overload of resources and may result from utilizing alternate facilities or additional resources often at a premium rate. Further, we assume that an additional cost c_w is incurred for each customer that waits for supply until the next period (e.g., in a distribution center, fulfillment does not occur until the next day or after the next delivery from the supplier.) This cost is analogous to a backorder cost. Let c_l be the cost of lost demand for those customers that choose not to wait. We let h be the holding cost for units at the end of the period. We assume $p_i + c_l - c_p - c_w \geq 0$ and $c_p \geq h$ so that production and inventory holding are profitable. We also assume that $c_w + h \geq c_M$ so that any units that could be delivered at the last stage (end of the period) are not delayed delivery for next period due to unrealistically high congestion costs.

Let $\vec{d}_{ij}$ be demand from class i waiting from a previous stage in stage j with $\vec{d}_j = \{\vec{d}_{1j}, \cdots, \vec{d}_{Kj}\}$. Let $\bar{d}_j = \{\vec{d}_j, d_j\}$. (Throughout the paper, we use the notational convention of symbols with an arrow, i.e., $\vec{}_{ij}$, refer to demand from class i waiting from previous stages, while symbols without an arrow refer to new demand from class i arriving in stage j. Symbols with a bar, i.e., $\bar{}_{ij}$, refer to all demands, i.e., both new and waiting demand.)

Let $\bar{D}_j = \sum_{i=1}^{K} \left(d_{ij} + \vec{d}_{ij}\right)$, the total new and waiting demand in stage j.

Let $D_j = \sum_{i=1}^{K} d_{ij}$ be the total new demand in stage j.

Let $\vec{D}_j = \sum_{i=1}^{K} \vec{d}_{ij}$ be the total waiting demand in stage j.

Let $D_{ij} = \sum_{k=i}^{K} d_{kj}$ be the new demand from classes i and higher in stage j.

Let $\vec{D}_{ij} = \sum_{k=i}^{K} \vec{d}_{kj}$ be the waiting demand from classes i and higher in stage j.

Let Ω_j be the set of all permutations of $\left(1, \cdots, \bar{D}_j\right)$ and let $\omega_j \in \Omega_j$ be some ordering of the demand, both new and waiting, in a stage j. Let D_{lij}^{ω} be the position in the ordering ω_j that the l^{th} customer from class i arriving in stage j is to be served in stage

j and let $\vec{D}^{\omega}_{lij}$ be the position in ω_j that the l^{th} demand from class i waiting from a previous stage is to be served in stage j. Then $\omega_j = \{D^{\omega}_{lij}, \vec{D}^{\omega}_{lij}\}$ for $i = 1,\cdots,K$ and $l = 1,\cdots,d_{ij}$ (or $\vec{d}_{ij}$) defines a priority order of service for the demand $\bar{D}_j$.

Let $\pi_{lij}\left(Y_j, z_j, \omega_j \middle| X_j, \bar{d}_j\right)$ be the margin obtained from the l^{th} unit of demand from new class i customers and let $\vec{\pi}_{lij}\left(Y_j, z_j, \omega_j \middle| X_j, \bar{d}_j\right)$ be the margin obtained from the l^{th} unit of demand from waiting class i customers for an allocation, discount and priority order given the inventory and demand. Then,

$$\pi_{lij}\left(Y_j, z_j, \omega_j \middle| X_j, \bar{d}_j\right) = \begin{cases} p_i & Y_j \geq D^{\omega}_{lij} \\ \gamma_{ij}\left(p_i - z_{ij} - c_p - c_w\right) - \left(1-\gamma_{ij}\right)c_l & Y_j < D^{\omega}_{lij} \end{cases} \tag{2.1}$$

and $$\vec{\pi}_{lij}\left(Y_j, z_j, \omega_j \middle| X_j, \bar{d}_j\right) = \begin{cases} c_p + c_w & Y_j \geq \vec{D}^{\omega}_{lij} \\ 0 & Y_j < \vec{D}^{\omega}_{lij} \end{cases}. \tag{2.2}$$

Let $\Pi_j\left(X_j, \bar{d}_j\right)$ be the optimal expected profit from stage j onward given inventory X_j and demand vector $\bar{d}_j$. The expected profit for stage j onward for a given Y_j, z_j and ω_j, is

$$\begin{aligned} &\pi_j\left(Y_j, z_j, \omega_j \middle| X_j, \bar{d}_j\right) \\ &= \sum_{i=1}^{K}\left[\sum_{l=1}^{d_{ij}} \pi_{lij}\left(Y_j, z_j, \omega_j \middle| X_j, \bar{d}_j\right) + \sum_{l=1}^{\vec{d}_{ij}} \vec{\pi}_{lij}\left(Y_j, z_j, \omega_j \middle| X_j, \bar{d}_j\right)\right] \\ &\quad - c_j\left(Y_j - g_j\right)^+ + E_{\bar{d}_{j+1}}\left[\Pi_{j+1}\left(X_{j+1}, \bar{d}_{j+1}\right)\right]. \end{aligned} \tag{2.3}$$

Observe a congestion cost is incurred for the units allocated over the capacity, g_i. (Note $(x)^+ = \max(0, x)$. Also let the indicator function, $1_x = 1$ if x is true, 0 otherwise.)

The problem the firm faces at stage j is

$$\Pi_j\left(X_j, \bar{d}_j\right) = \max_{Y_j, z_j, \omega_j}\left[\pi_j\left(Y_j, z_j, \omega_j \middle| X_j, \bar{d}_j\right)\right] \tag{2.4-a}$$

s.t.

$$\omega_j \in \Omega_j \tag{2.4-b}$$

$$Y_j \leq \min\left(X_j, \bar{D}_j\right) \tag{2.4-c}$$

$$X_{j+1} \leq X_j - Y_j \tag{2.4-d}$$

$$z_{ij} \leq \left(1-\alpha_{ij}\right)/\beta_{ij}\text{, for all } i \tag{2.4-e}$$

$$Y_j, z_{ij} \geq 0\text{, for all } i \tag{2.4-f}$$

We define the end of the period profit, $\Pi_{M+1}\left(X_{M+1}, \bar{d}_{M+1}\right) = \left(c_p - h\right) X_{M+1}$ to be the value from the units held to the next period, and note that we have already accounted for the expected revenues from waiting demand.

With no initial waiting demand, the profit for the multiple stage problem is

$$\max_{X \geq 0} \pi(X) = -c_p X + E\left[\Pi_1\left(X, d_1\right)\right] \tag{2.5}$$

We refer to the problem of determining the inventory allocation, the customer discounts and the prioritization of demand for all stages as the **Allocation, Discount and Prioritization (ADP) Problem.**

2.3. ADP Problem Solution

We solve the problem through dynamic programming starting first with the final stage M and then solving the problem by induction for stages $M-1,\cdots,1$.

2.3.1 Stage M

Determining the Discount z_{iM}^* For stage M, we have the following:

Proposition 2.1: *Given* $\left(X_M, \bar{d}_M\right)$*, the unique optimal price discount is*

$$z_{iM}^* = \max\left(\min\left[z_{iM}^{'}, (1-\alpha_{iM})/\beta_{iM}\right], 0\right) \tag{2.6}$$

where $z_{iM}^{'} = \dfrac{p_i + c_l - c_p - c_w}{2} - \dfrac{\alpha_{iM}}{2\beta_{iM}}$ *for* $i = 1,\ldots,K$. *(2.7)*

Proof. From (2.3), the first order conditions for the unconstrained problem

$$\max_{z_M}\left[\pi_M\left(Y_M, z_M, \omega_M \middle| X_M, \bar{d}_M\right)\right]$$

are

$$\begin{aligned}
\frac{\partial \pi_M}{\partial z_{iM}} &= \sum_{l=1}^{d_{iM}} \frac{\partial \pi_{liM}}{\partial z_{iM}} \\
&= \sum_{l:s.t.:Y_M < D_{liM}} \frac{\partial\left[\begin{array}{l}(\alpha_{iM} + \beta_{iM} z_{iM})(p_i - z_{iM} - c_p - c_w) \\ -(1-\alpha_{iM} - \beta_{iM} z_{iM}) c_l\end{array}\right]}{\partial z_{iM}} \qquad \text{for all } i = 1,\ldots,K. \\
&= \sum_{l:s.t.:Y_M < D_{liM}} \beta_{iM}\left(p_i + c_l - c_p - c_w\right) - \alpha_{iM} - 2\beta_{iM} z_{iM} \\
&= 0
\end{aligned}$$

which gives $z_{iM}^{'}$ as above. Constraining the solution so that $0 \le \gamma_{iM} \le 1$ provides the result. ■

Let $\gamma_{iM}^{*} = \alpha_{iM} + \beta_{iM} z_{iM}^{*}$. Observe that the discount z_{iM}^{*} offered does not depend on X_M, Y_M, $\bar{d}_M$ or ω_M. This allows us to determine ω_M and Y_M independently below.

Priority Determination ω_M^{*} and Allocation Determination Y_M^{*}. We define the (per unit) **demand denial penalties,** L_{iM} and $\vec{L}_M$, to be the reduction in profit if a demand from class i or one of the waiting demands, respectively, cannot be satisfied in stage M. Following the development in (2.1) and (2.2):

$$\begin{aligned} &L_{iM}\left(z_{iM}^{*}, \omega_M \middle| Y_M, \bar{d}_M\right) \\ &= p_i - \gamma_{iM}^{*}\left(p_i - z_{iM}^{*} - c_p - c_w\right) + \left(1 - \gamma_{iM}^{*}\right) c_l \\ &= p_i + c_l - \gamma_{iM}^{*}\left(p_i + c_l - z_{iM}^{*} - c_p - c_w\right) \end{aligned} \tag{2.8}$$

and

$$\vec{L}_M\left(z_{iM}^{*}, \omega_M \middle| Y_M, \bar{d}_M\right) = c_p + c_w. \tag{2.9}$$

We make the following claim:

Claim 2.1: *A priority sequence in stage M is optimal for all allocations Y_M if and only if the demand denial penalties are non-increasing with the order of the priority sequence.*

Proof. Assume ω_M' is an optimal priority sequence. Consider a pair of demands with $D_{l'i'M} = D_{liM} + 1$, that is, the l^{th} customer of class i is followed by the l'^{th} customer of class i' in ω_M'. (For purposes of presentation, consider the set of waiting customers just another class.) Suppose that $L_{i'} > L_i$ and $Y_M = D_{liM}$. Then the profit of the sequence can be improved by interchanging the two in the demand sequence which is a contradiction. Similarly, suppose $L_{i'} \le L_i$ if $D_{l'i'M} > D_{liM}$ for all i, i', l, l'. Then choosing any set of demands and rearranging them cannot decrease the cost for any Y_M, therefore, ω_M with a denied demand penalty non-increasing is optimal. ■

Based on the claim, we have the following:

Proposition 2.2: *A priority sequence with customers in* d_M *served in class order, from* K *to 1 prior to serving any waiting demand,* $\vec{d}_M$*, is optimal.*

Proof. Observe that $L_{i'} \geq L_i$ if $i' > i$ since

$$\begin{aligned} L_{i'M} - L_{iM} &= \left(p_{i'} + c_l - \gamma_{i'M}^* \left(p_{i'} + c_l - z_{i'M}^* - c_p - c_w\right)\right) - \\ &\quad \left(p_i + c_l - \gamma_{iM}^* \left(p_i + c_l - z_{iM}^* - c_p - c_w\right)\right) \\ &\geq \left(p_{i'} + c_l - \gamma_{i'M}^* \left(p_{i'} + c_l - z_{i'M}^* - c_p - c_w\right)\right) - \\ &\quad \left(p_i + c_l - \gamma_{iM} \left(p_i + c_l - z_{iM} - c_p - c_w\right)\right) \end{aligned}$$

since z_{iM}^* maximizes $\gamma_{iM}(z)\left(p_i + c_l - z - c_p - c_w\right)$ conditional on $0 \leq \gamma_{iM} = \alpha_{iM} + \beta_{iM} z_{iM} \leq 1$ (c.p. Proposition 2.1).

Letting $z_{iM} = z_{iM}'' \equiv min\left(z_{i'M}^*, (1-\alpha_{iM})/\beta_{iM}\right)$ implies $\gamma_{iM}'' \equiv min\left(\alpha_{iM} + \beta_{iM} z_{i'M}^*, 1\right)$. Since $\alpha_{iM} \geq \alpha_{i'M}$ and $\beta_{iM} \geq \beta_{i'M}$ and $\gamma_{i'M}^* \leq 1$, $\gamma_{iM}'' \geq \alpha_{i'M} + \beta_{i'M} z_{i'M}^* = \gamma_{i'M}^*$, and $p_{i'} + c_l - c_p - c_w \geq z_{i'M}^*$ from Proposition 2.1,

$$\begin{aligned} L_{i'M} - L_{iM} &\geq p_{i'} - p_i + \gamma_{iM}'' \left(\left(p_i - z_{iM}''\right) - \left(p_{i'} - z_{i'M}^*\right)\right) + \\ &\quad \left(\gamma_{iM}'' - \gamma_{iM}^*\right)\left(p_{i'} + c_l - z_{i'M}^* - c_p - c_w\right) \\ &\geq 0 \end{aligned}$$

Also observe

$$L_{iM} - \vec{L}_M \geq L_{1M} - \vec{L}_M = p_1 + c_l - \gamma_{1M}^* \left(p_1 + c_l - z_{1M}^* - c_p - c_w\right) - \left(c_p + c_w\right) \geq 0 .$$

Therefore ω^* serves new customers in class order prior to serving any waiting customers. ■

Proposition 2.3: *Allocating the inventory to all of the demand at the end of period, i.e.* $Y_M^* = \min\left(X_M, \bar{D}_M\right)$*, is optimal.*

Proof. The result follows directly from the assumption $c_w + h \geq c_M$. ■

Concavity of $E_{\vec{d}_M}\left[\Pi_M\left(X_M,\vec{d}_M\right)\right]$

In order to show the analogous results for the general stage j, we need to prove several monotonicity properties hold. To do so we introduce the following notation:

Let $\frac{\Delta\Pi_j\left(X_j,\vec{d}_j\right)}{\Delta X_j}$ be the change in the profit in stage j if an additional unit of inventory is available for allocation in stage j, i.e., if X_j+1 unit are available rather than X_j units. Let $\frac{\Delta\Pi_j\left(X_j,\vec{d}_j\right)}{\Delta\vec{D}_j}$ be the change in profit if an additional customer is waiting in stage j, i.e., if the number waiting is $\vec{D}_j+1$ rather than $\vec{D}_j$.

Note from (2.3), $\frac{\Delta\Pi_{M+1}\left(X_{M+1},\vec{d}_{M+1}\right)}{\Delta X_{M+1}}=c_p-h$ and also note from the discussion above, $\frac{\Delta\Pi_{M+1}\left(X_{M+1},\vec{d}_{M+1}\right)}{\Delta\vec{D}_{M+1}}=0$.

Proposition 2.4: *a)* $E_{\vec{d}_M}\left[\Delta\Pi_M/\Delta X_M\right]$ *is decreasing in* X_M, *increasing in* $\vec{D}_M$, *continuous and* $E_{\vec{d}_M}\left[\Delta\Pi_M/\Delta X_M\right]\geq c_p-h$.

b) $E_{\vec{d}_M}\left[\Delta\Pi_M/\Delta\vec{D}_M\right]$ *is decreasing in* $\vec{D}_M$, *increasing in* X_M, *continuous and* $c_p+c_w\geq E_{\vec{d}_M}\left[\Delta\Pi_M/\Delta\vec{D}_M\right]\geq 0$.

Proof. The profit rate function is

$$\frac{\Delta\Pi_M\left(X_M,\bar{d}_M\right)}{\Delta X_M} \equiv \frac{\Delta\pi_M\left(Y_M^*,z_M^*,\omega_M^*\middle|X_M,\bar{d}_M\right)}{\Delta X_M}$$

$$=\sum_{i=1}^{K}\left\{\sum_{l=1}^{d_{iM}}\frac{\Delta\pi_{liM}}{\Delta X_M}+\sum_{l=1}^{\bar{d}_{iM}}\frac{\Delta\vec{\pi}_{liM}}{\Delta X_M}\right\}$$

$$+\frac{\Delta\left[c_M\left(\min\left(X_M,\bar{D}_M\right)-g_M\right)^+\right]}{\Delta X_M}+\frac{\Delta\left[\left(X_M-\bar{D}_M\right)^+\left(c_p-h\right)\right]}{\Delta X_M}$$

$$=\begin{cases} L_{KM}\left(z_{iM}^*,\omega_M^*\middle|X_M,\bar{d}_M\right)-\left(c_M 1_{X_M\ge g_M}\right) & 0\le X_M<D_{KM} \\ \vdots & \vdots \\ L_{1M}\left(z_{1M}^*,\omega_M^*\middle|X_M,\bar{d}_M\right)-\left(c_M 1_{X_M\ge g_M}\right) & D_{2M}\le X_M<D_{1M} \\ c_p+c_w-\left(c_M 1_{X_M\ge g_M}\right) & D_{1M}\le X_M<D_M+\vec{D}_M \\ c_p-h & D_M+\vec{D}_M\le X_M \end{cases}$$

So that $\Delta\Pi_M/\Delta X_M \ge c_p-h$ and $\Delta\Pi_M/\Delta X_M$ is decreasing in X_M from (2.8), (2.9) and the assumption $c_w+h\ge c_M$. Under the assumption that the demand is continuous, taking expectations provides the result in 1).

A similar approach provides the results in 2) as

$$E\left[\Delta\Pi_M\left(X_M,\bar{d}_M\right)/\Delta\bar{D}_M\right]=(h+c_w)P\left(X_M>\bar{D}_M\right)-c_M P\left(X_M>\bar{D}_M>g_M\right),$$

and by observing that the absolute value of the partial derivative of the second term is dominated by that of the first term. ■

2.3.2. Stage j

We now consider the problem faced at stage j. We show by induction how the discounts to offer, z_{ij}, the priority to assign to customers in allocating inventory, ω_j, and the amount of inventory to allocate, Y_j, are found. We show that, as in the stage M solution, the discounts to offer can be found by comparing the incremental revenue received after price discounting, though now an algorithm is required to find a feasible optimal solution. We then show that the optimal priority sequence is independent of these discounts and orders new customers first (in class order) followed by waiting demand as in the stage M solution. Finally we show how to determine the amount of inventory to allocate in stage j.

We assume for the induction that for $j'>j$, $E\left[\Pi_{j'}\left(X_{j'},\bar{d}_{j'}\right)\right]$ is concave in $X_{j'}$ and $\vec{D}_{j'}$, that $E\left[\Delta\Pi_{j'}/\Delta X_{j'}\right]$ is increasing in $\vec{D}_{j'}$, continuous and $E\left[\Delta\Pi_{j'}/\Delta X_{j'}\right]\geq c_p-h$ and that $E_{\bar{d}_{j'}}\left[\Delta\Pi_{j'}/\Delta\vec{D}_{j'}\right]$ is increasing in $X_{j'}$, continuous and $c_p+c_w\geq E_{\bar{d}_{j'}}\left[\Delta\Pi_{j'}/\Delta\vec{D}_{j'}\right]\geq 0$. Note that this implies that holding the remaining units in inventory is profitable and, without loss of generality, we have $X_{j+1}=X_j-Y_j\geq 0$ in the rest of our paper.

Determining the Discount z_{ij}^* Let

$$z_{ij}^{'}=\frac{p_i+c_l-c_p-c_w}{2}-\frac{\alpha_{ij}}{2\beta_{ij}}+\frac{1}{2}E_{\bar{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}\left(X_j-Y_j,\bar{d}_{j+1}\right)}{\Delta\vec{D}_{j+1}}\right]. \tag{2.10}$$

Proposition 2.5: *In stage j, $z_{ij}^{'}$ maximizes $\pi_j\left(Y_j,z_j,\omega_j\middle|X_j,\bar{d}_j\right)$.*

Proof.

$$\begin{aligned}
&\frac{\partial\pi_j}{\partial z_{ij}}\left(Y_j,z_j,\omega_j\middle|X_j,\bar{d}_j\right)\\
&=\frac{\partial}{\partial z_{ij}}\sum_{l=1}^{d_{ij}}\pi_{lij}\left(Y_j,z_j,\omega_j\middle|X_j,\bar{d}_j\right)+\frac{\partial}{\partial z_{ij}}E_{\bar{d}_{j+1}}\left[\Pi_{j+1}\left(X_j-Y_j,\bar{d}_{j+1}\right)\right]\\
&=\left(\beta_{ij}\left(p_i+c_l-z_{ij}-c_p-c_w\right)-\gamma_{ij}\right)c_{ij}+E_{\bar{d}_{j+1}}\left[\frac{\partial\vec{D}_{j+1}}{\partial z_{ij}}\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}}\left(X_j-Y_j,\bar{d}_{j+1}\right)\right]
\end{aligned}$$

where c_{ij} is the number of customers of class i denied service in stage j. Letting $\gamma_{ij}=\alpha_{ij}+\beta_{ij}z_{ij}$ and observing $\partial\vec{D}_{j+1}/\partial z_{ij}=\beta_{ij}c_{ij}$ implies

$$\frac{\partial\pi_j}{\partial z_{ij}}=c_{ij}\begin{pmatrix}-\alpha_{ij}-2\beta_{ij}z_{ij}+\beta_{ij}\left(p_i+c_l-c_p-c_w\right)\\+\beta_{ij}E_{\bar{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}}\left(X_j-Y_j,\bar{d}_{j+1}\right)\right]\end{pmatrix} \tag{2.11}$$

Noting $\partial\pi_j/\partial z_{ij}$ is decreasing in z_{ij} since $\vec{D}_{j+1}$ increasing in z_{ij} and $E_{\bar{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta\vec{D}_{j+1}\right]$ decreasing in $\vec{D}_{j+1}$ from the induction assumption, setting $\partial\pi_j/\partial z_{ij}=0$ and solving provides the result. ■

By comparing (2.7) with (2.10) we observe that the optimal discount maximizes the current profit plus a term that reflects the change in future profits resulting from a change in the total number of waiting customers.

Noting that $z_{ij}^{'}$ solves the unconstrained problem, the constrained problem, given Y_j, X_j, ω_j and $\bar{d}_j$, is:

$$\max\ \pi_j\left(Y_j,z_j,\omega_j\middle|X_j,\bar{d}_j\right)\quad \text{s.t.}\quad 0\le z_{ij}\le\left(1-\alpha_{ij}\right)/\beta_{ij}::\forall i,j, \qquad (2.12)$$

We have the following proposition:

Proposition 2.6: *Suppose in an optimal solution, z^*, to (2.12) for $x,y\in\{1,\cdots,K\}$ that* $0<z_{xj}<\dfrac{1-\alpha_{xj}}{\beta_{xj}}$ *and* $0<z_{yj}<\dfrac{1-\alpha_{yj}}{\beta_{yj}}$, *then*

$$\Delta z_{xy,j}\equiv z_{xj}^*-z_{yj}^*=\frac{1}{2}\left[\left(p_x-\frac{\alpha_{xj}}{\beta_{xj}}\right)-\left(p_y-\frac{\alpha_{yj}}{\beta_{yj}}\right)\right].$$

Proof. Consider the Kuhn-Tucker conditions for Problem (2.12), if $0\le z_{xj}^{'}\le\dfrac{1-\alpha_{xj}}{\beta_{xj}}$ and $0\le z_{yj}^{'}\le\dfrac{1-\alpha_{yj}}{\beta_{yj}}$, $\dfrac{\partial\pi_j}{\partial z_{xj}}=\dfrac{\partial\pi_j}{\partial z_{yj}}=0$. Then $\dfrac{1}{\beta_{xj}c_{xj}}\dfrac{\partial\pi_j}{\partial z_{xj}}-\dfrac{1}{\beta_{yj}c_{yj}}\dfrac{\partial\pi_j}{\partial z_{yj}}=0$ implies, by (2.11) $\left(-2z_{xj}+p_x-\dfrac{\alpha_{xj}}{\beta_{xj}}\right)-\left(-2z_{yj}+p_y-\dfrac{\alpha_{yj}}{\beta_{yj}}\right)=0$ which gives the result. ■

The proposition implies that if we sort the classes by $p_i-\alpha_{ij}/\beta_{ij}$ and increase z_{ij} from 0 for each class in this order, while maintaining a separation of Δz_{ij} and ensuring that no z_{ij} exceeds its upper bound, we will eventually find an optimal solution to the constrained problem. We formally define this through the following algorithm:

Algorithm 2.1: Discount z_{ij}^* Determination Algorithm

Initialization

Let $r_i = p_i - \alpha_i/\beta_i$ and $b_i = (1-\alpha_i)/\beta_i$. (Throughout the algorithm we suppress the stage subscript j for ease of presentation)

Order the classes by r_i. (For purposes of clarity, we assume without loss of generality, that $r_1 \geq \cdots \geq r_K$.) Set $z_i = 0 : \forall i$. Set $x = 1$ (x is the last class entering the active set, i.e. the set of classes for which z_i is not at its upper or lower bounds). Let $\mathrm{A} = \{1\}$, be the set of customer classes that are active (including a class entering the active set), Set $\Delta z_x \equiv (r_x - r_{x+1})/2$ for $1 \leq x < K$ and $\Delta z_K = \infty$.

Step 1. Repeat $A := A \cup \{x+1\}$, $x := x+1$ while $\Delta z_x = 0$. Set $\Delta b = \min_{y \in A}(b_y - z_y)$. Let $B = \{y | b_y - z_y = \Delta b\}$, be the set of active classes that would next achieve their upper bounds. Set $\Delta z = \min(\Delta b, \Delta z_x)$.

Step 2. Test: Is there $\Delta z' \in (0, \Delta z)$ such that by letting $z_y' = z_y + \Delta z' : \forall : y \in A$, z_y' solves (2.10) for one class in A. If so, let $z_y := z_y + \Delta z' : \forall : y \in A$ and stop. (The solution is optimal at the current levels of z_y.) Otherwise, go to Step 3.

Step 3. $\Delta z_x := \Delta z_x - \Delta b$. $z_y := z_y + \Delta z : \forall : y \in A$. $A := A \setminus B$ if $\Delta z := \Delta b$.

Step 4. If $A = \varnothing$ and $x = K$, stop. (All z_x are at their upper bounds.)

Step 5. If $A \neq \varnothing$ and $\Delta z_x > 0$, go to Step 1.

Step 6. $A := A \cup \{x+1\}$. $x := x+1$. Go to step 1. ■

In the algorithm the value of z_i is successively increased from 0 until either the value which is active (i.e., in set A) solves the first order condition or reaches its upper bound. Since a spacing of Δz_x is maintained between active classes, Proposition 2.5 holds for active classes. In each iteration either a class enters or at least one class exits the active set, so that there are at most $2K$ iterations. In each iteration the number of tests (in Step 2) is $O(\log \Delta z)$ since $\partial \pi_j / \partial z_{ij}$ decreases in z_{ij} from Proposition 2.5. This results in the following:

Proposition 2.7: *In stage j, given Y_j, X_j, ω_j and $\bar{d}_j$, the optimal price discounts vector z_j^* is unique and given by the Discount Determination Algorithm.* ■

Determining ω_j^* Following the development for the stage M case, we define the (per unit) demand denial penalties based on (2.1), (2.2) and (2.3)

$$L_{ij}\left(Y_j, z_{ij}^*, \omega_j \middle| X_j, \bar{d}_j\right)$$
$$= p_i + c_l - \gamma_{ij}^*\left(p_i + c_l - z_{ij}^* - c_p - c_w\right) - \gamma_{ij}^* E_{\bar{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}}\left(X_j - Y_j, \bar{d}_{j+1}\right)\right] \tag{2.13}$$

and

$$\vec{L}_j\left(Y_j, z_{ij}^*, \omega_j \middle| X_j, \bar{d}_j\right) = c_p + c_w - E_{\bar{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}}\left(X_j - Y_j, \bar{d}_{j+1}\right)\right]. \tag{2.14}$$

Claim 2: *L_{ij} and $\vec{L}_{ij}$ are continuous, increasing in $\vec{D}_j$ and decreasing in X_j.*

Proof. We observe the results from (2.13), (2.14) and the induction assumptions $E_{\bar{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta\vec{D}_{j+1}\right]$ continuous, decreasing in $\vec{D}_{j+1}$ and increasing in X_j. ■

Claim 2.3: *A priority sequence ω_j is optimal for all allocations Y_j if and only if the demand denial penalty in stage j is non-increasing with the order of the priority sequence.*

Proof. Similar to Claim 2.1. ■

Proposition 2.8: A priority sequence with customers in d_j served in class order, from K to 1, prior to serving a waiting demand, $\vec{d}_j$, is optimal.

Proof. Let $\vec{d}^*_{j+1}$ denote the (random) total demand vector in stage $j+1$ under $z_j = z^*_j$. Observe that $L_{i'j} \geq L_{ij}$ if $i' > i$ since (suppressing the subscript j for conciseness)

$$L_{i'} - L_i$$
$$\geq \left(p_{i'} - \gamma^*_{i'} \left(p_{i'} + c_l - z^*_{i'} - c_p - c_w + E_{\vec{d}_{j+1}} \left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}} \left(X_j - Y_j, \vec{d}^*_{j+1} \right) \right] \right) \right) -$$
$$\left(p_i - \gamma_i \left(p_i + c_l - z_i - c_p - c_w + E_{\vec{d}_{j+1}} \left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}} \left(X_j - Y_j, \vec{d}^*_{j+1} \right) \right] \right) \right)$$

for $\gamma_i = \alpha_i + \beta_i z_i$ for any z_i such that $0 \leq z_i \leq (1-\alpha_i)/\beta_i$ since $z_i^{'}$ maximizes $\gamma_i(z) \left(p_i + c_l - z - c_p - c_w + E_{\vec{d}_{j+1}} \left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}} \left(X_j - Y_j, \vec{d}^*_{j+1} \right) \right] \right)$ and z^*_i equals either $z_i^{'}$ or 0 or $(1-\alpha_i)/\beta_i$ from the monotonicity of $E_{\vec{d}_{j+1}} \left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}} \left(X_j - Y_j, \vec{d}^*_{j+1} \right) \right]$.

Letting $z_i^{''} \equiv min\left(z^*_{i'}, (1-\alpha_i)/\beta_i\right)$ implies $\gamma_i^{''} \equiv min\left(\alpha_i + \beta_i z^*_{i'}, 1\right) \geq \alpha_{i'} + \beta_{i'} z^*_{i'} = \gamma^*_{i'}$ since $\alpha_i \geq \alpha_{i'}$ and $\beta_i \geq \beta_{i'}$. Therefore

$$L_{i'} - L_i \geq \left(1 - \gamma_i^{''}\right)\left(p_{i'} - p_i\right) + \gamma_i^{''}\left(z^*_{i'} - z_i^{''}\right)$$
$$+ \left(\gamma_i^{''} - \gamma^*_{i'}\right) \left(p_{i'} + c_l - z^*_{i'} - c_p - c_w + E_{\vec{d}_{j+1}} \left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}} \left(X_j - Y_j, \vec{d}^*_{j+1} \right) \right] \right)$$
$$\geq 0$$

since $z^*_{i'} \leq \left(p_{i'} + c_l - c_p - c_w + E_{\vec{d}_{j+1}} \left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}} \left(X_j - Y_j, \vec{d}^*_{j+1} \right) \right] \right)^+$ by Proposition 2.7 and $p_{i'} + c_l - c_p - c_w + E_{\vec{d}_{j+1}} \left[\Delta\Pi_{j+1} / \Delta\vec{D}_{j+1} \right] \geq 0$ from the induction assumption $E_{\vec{d}_{j+1}} \left[\Delta\Pi_{j+1} / \Delta\vec{D}_{j+1} \right] \geq 0$ and the assumption $p_{i'} + c_l - c_p - c_w \geq 0$. ■

Determining Y_j^* Let Y_{ij} be the number of units to allocate in stage j if when the last customer to receive a unit is from class i. Similarly, let $\vec{Y}_j$ be the number of units allocated in stage j when the last customer to receive a unit is among the waiting customers. We have the following two claims regarding these values:

Claim 2.4: Y_{ij}^* is unique and solves

$$
\begin{aligned}
&\frac{\Delta\pi_j}{\Delta Y_{ij}}\left(Y_{ij}, z_j^*, \omega_j^* \middle| X_j, \bar{d}_j\right) \\
&= p_i + c_l - \gamma_{ij}^*\left(p_i + c_l - z_{ij}^* - c_p - c_w + E_{\bar{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}}\left(X_j - Y_j, \bar{d}_{j+1}\right)\right]\right) \\
&\quad -\left(c_j 1_{Y_j \ge g_j}\right) - E_{\bar{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta X_{j+1}}\left(X_j - Y_j, \bar{d}_{j+1}\right)\right] \\
&= L_{ij} - \left(c_j 1_{Y_j \ge g_j}\right) - E_{\bar{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta X_{j+1}}\left(X_j - Y_j, \bar{d}_{j+1}\right)\right] \\
&= 0
\end{aligned}
$$

and $\vec{Y}_j^*$ is unique and solves

$$
\frac{\Delta\vec{\pi}_j}{\Delta\vec{Y}_j}\left(\vec{Y}_j, z_j^*, \omega_j^* \middle| X_j, \bar{d}_j\right) = \vec{L}_j - \left(c_j 1_{Y_j \ge g_j}\right) - E_{\bar{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta X_{j+1}}\left(X_j - Y_j, \bar{d}_{j+1}\right)\right] = 0
$$

Proof. $\frac{\Delta\pi_j}{\Delta Y_{ij}}\left(Y_{ij}, z_j^*, \omega_j^* \middle| X_j, \bar{d}_j\right)$ is the marginal profit when a unit in X_j is shifted from stage $j+1$ to stage j and is used to satisfy a demand of class i. Considered as two operational steps, 1) increasing an unit on X_j and allocating it to a new demand from class i at stage j, the marginal profit is $L_{ij} - \left(c_j 1_{Y_j \ge g_j}\right)$. 2) reducing a unit from X_j and X_{j+1}, the marginal profit is $-E_{\bar{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta X_{j+1}\right]$. By the induction assumption, both $E_{\bar{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta X_{j+1}\right]$ and $E_{\bar{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta\vec{D}_{j+1}\right]$ are monotonic and so the solution exists and by continuity of the demand distribution, unique. A similar proof holds for the case that an unit in X_j is shifted from stage $j+1$ to stage j and used to serve a waiting demand in $\vec{Y}_j$. ■

Claim 2.5: *Y_{ij} is non-decreasing in i and $\vec{Y}_j \le Y_{1j}$.*

Proof. $\frac{\Delta\Pi_j}{\Delta Y_{i+1,j}} - \frac{\Delta\Pi_j}{\Delta Y_{ij}} = L_{i+1,j} - L_{ij} \ge 0$ from the proof of Proposition 2.8 so that $\Delta\Pi_j / \Delta Y_{ij}$ is non-decreasing in i. Therefore the solution of $\Delta\Pi_j / \Delta Y_{ij} = 0$ is also non-decreasing in i. ■

The actual allocation in stage j follows the following **inventory allocation rule**: Y_j equals Y_{ij} if the Y_{ij}^{th} customer is of class i, i.e., $D_{lij} = Y_{ij}$ for some $l \in \{1, \cdots, d_{ij}\}$ and equals $\vec{Y}_j$ if the $\vec{Y}_j^{th}$ customer is among the waiting demand, i.e. $\vec{D}_{lij} = \vec{Y}_{ij}$ for some i and l. Finally, if $\bar{D}_j < \vec{Y}_j$, i.e., the total demand is less than the optimal amount to allocate for all the waiting demand, then Y_j equals $\bar{D}_j$. In all cases, Y_j must be less than or equal to X_j.

Since the optimal priority sequence is to serve the new customers in class order from K to 1, prior to serving any of the waiting customers, we observe that D_{lij} is decreasing in i and so in light of Claim 2.5 we have the following proposition:

Proposition 2.9: *The optimal number of units allocated in stage j is unique and is given by the inventory allocation rule.*

Proof. Suppose that $D_{lij} = Y_{ij}$ for $l \in \{1, \cdots, d_{ij}\}$ and $Y_{ij} \le X_j$. Then all of the new demand from classes $i+1, \cdots, K$ are allocated inventory and no demand from classes $1, \cdots, i-1$ or waiting demand is allocated inventory in accordance with the priority sequence. By Claim 2.5 $Y_{i'j} \ge Y_{ij}$ and by Proposition 2.8, $D_{l'i'j} < D_{lij}$ for $i' > i$ and $l' \in \{1, \cdots, d_{i'j}\}$.

Similarly, for classes $i' < i$ and all waiting demand, we know $Y_{i'j} \le Y_{ij}$ and $D_{l'i'j} > D_{lij}$. Therefore $D_{l'i'j} \ne Y_{i'j}$. Therefore $Y_j^* = Y_{ij}$. A similar argument follows if $\vec{D}_{lij} = \vec{Y}_j$ and $\vec{Y}_j \le X_j$ so $Y_j^* = \vec{Y}_j$. Finally if, $\bar{D} < \vec{Y}_j$ and $\bar{D} \le X_j$, then $D_{lij} \le \bar{D} < \vec{Y}_j \le Y_{ij}$ for all i and $\vec{D}_{lij} \le \bar{D} < \vec{Y}_j$ for all i so that letting $Y_j^* = \bar{D}_j$ is feasible and optimal, since holding the remained units is profitable from the induction assumption. In any of these cases, if $X_j < Y_j$ then from (2.4-c) we know $Y_j^* = X_j^*$. ■

Having established z_{ij}^*, ω_j^* and Y_j^*, we need to show that the induction assumptions hold.

Proposition 2.10: *a).* $E_{\bar{d}_j}\left[\Delta\Pi_j / \Delta X_j\right]$ *is decreasing in* X_j, *increasing in* $\vec{D}_j$, *continuous and* $E_{\bar{d}_j}\left[\Delta\Pi_j / \Delta X_j\right] \ge c_p - h$.

b). $E_{\bar{d}_j}\left[\Delta\Pi_j / \Delta\vec{D}_j\right]$ *is decreasing in* $\vec{D}_j$, *increasing in* X_j, *continuous and* $c_p + c_w \ge E_{\bar{d}_j}\left[\Delta\Pi_j / \Delta\vec{D}_j\right] \ge 0$.

Proof. If $X_j < \bar{D}_j$, the incremental inventory Δ may be allocated at stage j and

$$\lim_{\Delta \to 0} \frac{\pi_j\left(Y_j^* + \Delta, z_j^*, \omega_j^* \middle| X_j + \Delta, \bar{d}_j\right) - \pi_j\left(Y_j^*, z_j^*, \omega_j^* \middle| X_j, \bar{d}_j\right)}{\Delta}$$

$$= \sum_{i=1}^{K} \sum_{l=1}^{d_{ij}} \left(p_i + c_l - \gamma_{ij}^* \begin{pmatrix} p_i + c_l - z_{ij}^* - c_p - c_w \\ + E_{\bar{d}_{j+1}} \left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}} \left(X_j - Y_j, \bar{d}_{j+1} \right) \right] \end{pmatrix} \right) \cdot 1_{Y_j^* = D_{lij}^{\omega^*}}$$

$$+ \sum_{i=1}^{K} \sum_{l=1}^{\vec{d}_{ij}} \left(c_p + c_w - E_{\bar{d}_{j+1}} \left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}} \left(X_j - Y_j, \bar{d}_{j+1} \right) \right] \right) \cdot 1_{Y_j^* = \vec{D}_{lij}^{\omega^*}} - \left(c_j 1_{g_j \le Y_j^* \le \bar{D}_j} \right)$$

$$= \sum_{i=1}^{K} L_{ij} \cdot 1_{D_{i+1,j} \le Y_j^* < D_{ij}} + \vec{L}_j \cdot 1_{D_j \le Y_j^* < \bar{D}_j} - \left(c_j 1_{g_j \le Y_j^* \le \bar{D}_j} \right)$$

If the incremental inventory Δ is allocated at stage $j+1$, then

$$\Delta\pi_j\left(X_j,\vec{d}_j\right)/\Delta X_j = E_{\vec{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta X_{j+1}\right]$$

Since we always allocate the units to the stage with higher incremental profit, then from the induction assumption, it must be that

$$\frac{\Delta\pi_j}{\Delta X_j} = \max\left\{\left[\sum_{i=1}^{K} L_{ij}\cdot 1_{D_{i+1,j}\le Y_j^*<D_{ij}} + \vec{L}_j\cdot 1_{D_j\le Y_j^*<\vec{D}_j} - \left(c_j 1_{g_j\le Y_j^*\le \vec{D}_j}\right)\right], E_{\vec{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta X_{j+1}}\right]\right\}$$

$$\ge E_{\vec{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta X_{j+1}}\right] \ge c_p - h \ge 0$$

Since L_{ij}, $\vec{L}_j$ and $E_{\vec{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta X_{j+1}\right]$ are decreasing in X_j, increasing in $\vec{D}_j$, and continuous from Claim 2.2 and the induction assumption, by taking expectations the result in a) holds.

A similar approach provides the results in b) by

$$\frac{\Delta\Pi_j}{\Delta\vec{D}_j}\left(X_j,\vec{d}_j\right) = \max\left\{c_p + c_w - E_{\vec{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta X_{j+1}}\right] - \left(c_j 1_{g_j\le\vec{D}_j}\right), E_{\vec{d}_{j+1}}\left[\frac{\Delta\Pi_{j+1}}{\Delta\vec{D}_{j+1}}\right]\right\}$$ ■

Let X^* be the optimal base-stock held. We have:

Claim 2.6: *X^* is unique and solves* $\Delta\pi(X)/\Delta X = -c_p + E_{d_1}\left[\Delta\Pi_1(X,d_1)/\Delta X_1\right] = 0$

Proof. The result holds from (2.5) and Proposition 2.10 b). ■

We summarize the results in this section as follows:

Theorem 2.1. For the Allocation, Discount and Prioritization (ADP) Problem, the optimal price discounts, the optimal priority sequence, the optimal unit allocation policy and the optimal base stock level are unique. ■

2.3.3 Expected Demand Heuristic

The computational time for the optimal ADP solution increases exponentially with the number of the stages and the customer classes. In this section, we introduce the expected demand heuristic (EDH) which assumes demand equals its expectation in each stage when we determine the price discount and allocation rule.

EDH First, find the optimal price discount at stage M, i.e. z_M^*, using (2.6). Let $Y_{M-1}^e\left(X_{M-1},\vec{d}_{M-1}\right)$ and $z_{M-1}^e\left(X_{M-1},\vec{d}_{M-1}\right)$ denote the solution to the ADP problem at stage $M-1$ given $\left(X_{M-1},\vec{d}_{M-1}\right)$ assuming the arriving new demand equals its expectation, i.e., $d_{M-1}=E\left(d_{M-1}\right)=\left(E\left(d_{1,M-1}\right),\cdots,E\left(d_{K,M-1}\right)\right)$. We can determine $\left(Y_j^e,z_j^e\right)$ by induction for each stage j given $\left(X_j,\vec{d}_j\right)$ and assuming the arriving new demand $d_j=E\left(d_j\right)$. We can find the base-stock level X^e under the policy $\left(Y_j^e,z_j^e\right)\equiv\left(Y_1^e,\cdots,Y_M^e,z_1^e,\cdots,z_M^e\right)$. Since we only calculate the price discounts and the allocation policy once at each stage, the computational time is reduced significantly. Moreover, since Y_j^e and z_j^e are constant matrices with two dimensions $\left(X_j,\vec{d}_j\right)$, the system can make real-time decisions if we store the values of $\left(Y^e,z^e\right)$.

We can show that the EDH is optimal when all demand is lost, i.e. $\alpha_{ij}=\beta_{ij}=0$ so that $\vec{D}_j=0$. In this case, the profit function at stage j is dependent only on the inventory level from the previous period and there is no need to offer a discount ($z_{ij}^*=0$). Further, the demand denial penalty, L_{ij}, is independent of the demand in the previous periods. Therefore, considering the following period, there exists a unique pair of inventory levels, denoted as $\left(X_{i,j+1}^{*1},X_{i,j+1}^{*2}\right)$, where $X_{i,j+1}^{*1}$ solves $E_{\vec{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta X_{j+1}\right]=L_{ij}$ and $X_{i,j+1}^{*2}$ solves $E_{\vec{d}_{j+1}}\left[\Delta\Pi_{j+1}/\Delta X_{j+1}\right]=L_{ij}-c_j$. Therefore, we can use a simplified allocation rule, called the **critical inventory level rule**, where class i in stage j receives the last unit allocated in stage j if and only if the on-hand inventory falls to $X_{i,j+1}^{*1}$ if $Y_j\le g_j$ or $X_{i,j+1}^{*2}$ if $Y_j>g_j$.

Proposition 2.11: *For the lost demand case, the EDH approach for calculating price discounts and the critical inventory level rule are optimal.*

Proof. Since the price discount is zero and there exists no waiting demand, the prioritized ordering is naturally from class K to 1. Noting $X_{i,j+1}^{*1} \geq X_{i,j+1}^{*2}$, then similar to the proof of Proposition 9, the optimal number of units allocated in stage j is unique and is given by the critical inventory level rule. Since the price discount and the allocation rule are identical for any observed demand, the solution of EDH is also optimal to the ADP problem. ■

For the general case, we can demonstrate the effectiveness of the EDH for a two-stage problem. Given (X, d_1), the price discount error is

$$\begin{aligned} \Delta z &= \left|z_{i1}^{*} - z_{i1}^{e}\right| \leq \left|z_{i1}^{'} - z_{i1}^{'e}\right| \\ &= \frac{h + c_w}{2} P\left[\max\left(\vec{D}_2, \vec{D}_2^e\right) \geq X - Y_1 - D_2 > \min\left(\vec{D}_2, \vec{D}_2^e\right)\right] \end{aligned} \quad (2.15)$$

where $z_{i1}^{'e}$ and $\vec{D}_2^e$ is the global optimal price discount and the waiting demand of EDH, respectively. The error on the profit rate function is

$$\begin{aligned} \Delta\left[\frac{\partial \Pi_1}{\partial Y_{i1}}\right] &= \left|\frac{\partial \pi_1}{\partial Y_{i1}}\left(Y_{i1}^e, z_1^e, \omega_1^* \middle| X, d_1\right) - \frac{\partial \pi_j}{\partial Y_{i1}}\left(Y_{i1}^*, z_1^*, \omega_1^* \middle| X, d_1\right)\right| \\ &= \left|\frac{\partial \pi_1}{\partial Y_{i1}}\left(Y_{i1}^e, z_1^e, \omega_1^* \middle| X, d_1\right) - \frac{\partial \pi_1}{\partial Y_{i1}}\left(Y_{i1}^e, z_1^e, \omega_1^* \middle| X, E(d_1)\right)\right| \\ &= \left(1 - r_{i1}^e\right)\left(h + c_w\right) \cdot P\begin{pmatrix} \max\left[\vec{D}_2^e \middle| d_1, \vec{D}_2^e \middle| E(d_1)\right] \\ \geq X - Y_{i1} - D_2 \\ > \min\left[\vec{D}_2^e \middle| d_1, \vec{D}_2^e \middle| E(d_1)\right] \end{pmatrix} \end{aligned} \quad (2.16)$$

Because the likelihood of demand waiting is based on the discounts which are calculated optimally in the EDH given the expected demand, the probabilities $P\left(D_2 \geq X - Y_1 - \vec{D}_2\right)$ and $P\left(D_2 \geq X - Y_1 - \vec{D}_2^e\right)$ are close to each other for most practical cases. Thus, the probability terms in (2.15) and (2.16) are very small. This implies that the heuristic is quite accurate, even though the exact error bound cannot be clearly established.

To illustrate the accuracy of the EDH heuristic, we present computational results in Table 1. We experiment with the following data set: $c_p = 1$, $h = 0.6$, $K = 2$, $\alpha_j = (0,0)$ and $p = (p_1, 2)$, $\beta_j = (\beta_1, 0.1)$, $j = 1,2$. All demands d_{ij}, $i = 1,2$ and $j = 1,2$, are independent with each other and follow the same normal distribution with $\mu = 300$ and $\sigma = 90$. We present results by varying the parameters p_1 and β_1. As it can be seen both the EDH expected profit and the initial inventory decision are very close (less than 2% deviation for the profit and less than 1% deviation for the initial inventory) to those of the optimal ADP solution. (The same results hold over larger numerical sets we experimented with, but not reported here for brevity of presentation reasons).

Parameters		Expected Profit			Base Stock		
p_1	β_1	ADP	EDH	Error	ADP	EDH	Error
1.2	5.0000	457.1375	457.1375	0.00%	954.6	954.6	0.00%
1.4	1.2500	504.9331	495.7363	1.82%	913.1	908.2	0.53%
1.6	0.3125	538.9647	538.7590	0.04%	965.8	961.9	0.40%
1.8	0.0781	616.9130	616.8974	0.00%	1174.8	1177.2	-0.21%
2.0	0.0195	719.8512	719.8504	0.00%	1232.4	1232.4	0.00%

Table 2.1: Accuracy of Expected Profit and Base Stock Level for EDH

2.4. A Continuous Time Heuristic

In most cases, demands arrive according to a continuous time stochastic process and firms are required to respond to demands immediately, either accepting the order or offering a discount to encourage the customer to wait for delivery in the next period (or, of course, rejecting the demand). We can approximate such a case by dividing the period into a large number of stages, M, of, say, equal duration. However, in making this approximation, we need to assume that the congestion cost, $c_j = 0$, because the capacity in each stage, g_j, becomes increasingly small as M increases. Therefore, with no congestion cost, it is easy to see that, in an informal sense, there is less information available when a decision is made regarding discounts. Because of this the profit for the firm would decrease as M increases.

Because of the computational difficulty in solving the continuous time problem optimally for a large number of stages, we consider a heuristic that determines whether a demand should be accepted based on the current inventory, waiting demand and the assumption that all future demands will occur in a single stage. That is, consider a single arrival from class i at t. Let $X(t)$ and $\vec{d}(t)$ denote the inventory level and the waiting demand at time t. Let $d_k(t,T]$ denote the random variable of demand from class k customers in the time interval $(t,T]$ for all classes. We consider the following two-stage heuristic (**TSH**):

Consider a two-stage ADP problem where the first stage has just been completed with $d_{i1} = 1$ and $d_{i'1} = 0$ for $i' \neq i$, $X_1 = X(t)$, $\vec{d}_1 = \vec{d}(t)$, $d_{i2} = d_i(t,T]$. That is, there is only the arrival from class i to consider for allocating the current inventory and the demand for the second stage is assumed to be distributed as the total of all future demands. Because the congestion cost is 0, the waiting demand is only allocated inventory at time T. Then we can calculate a price discount in TSH, denoted as $z_i^h(t)$, is according to (2.6) and (2.10) and Proposition 2.7.

$$z_i^h(t)=\max\left[\min\left(\left(\begin{array}{l}\frac{p_i+c_l-c_p-c_w}{2}-\frac{\alpha_i(t)}{2\beta_i(t)}\\+\frac{1}{2}(h+c_w)P\left(X(t)>\Sigma\vec{d}(t)+\Sigma d_k(t,T]\right)\end{array}\right),\frac{1-\alpha_i(t)}{\beta_i(t)}\right),0\right].$$

We then allocate a unit of $X(t)$ to the class-i customer if and only if $X(t)\geq X_i^*|t,\Sigma\vec{d}(t)$, where $X_i^*|t,\Sigma\vec{d}(t)$ is the unique solution of

$$\begin{aligned}&p_i+c_l-\left(\alpha_i(t)+\beta_i(t)z_i^h(t)\right)\left(\begin{array}{l}p_i+c_l-z_i^h(t)-c_p-c_w\\+(h+c_w)P\left(X_i>\Sigma\vec{d}(t)+\Sigma d_k(t,T]\right)\end{array}\right)\\&=\sum_{i=1}^{K}\left(\begin{array}{l}\left[p_i+c_l-\left(\alpha_i(T)+\beta_i(T)z_i^h(T)\right)\left(p_i+c_l-z_i^h(T)-c_p-c_w\right)\right]\\\cdot P\left(\Sigma_i^K d_k(t,T]\geq X_i>\Sigma_{i+1}^K d_k(t,T]\right)\end{array}\right)\\&\quad+\left(c_p+c_w\right)\cdot P\left(\Sigma d_k(t,T]+\Sigma\vec{d}(t)\geq X_i>\Sigma d_k(t,T]\right)\\&\quad+\left(c_p-h\right)\cdot P\left(X_i>\Sigma d_k(t,T]+\Sigma\vec{d}(t)\right),\end{aligned}$$

as shown by Lemma 4 where

$$z_i^h(T)=\max\left[\min\left(\frac{p_i+c_l-c_p-c_w}{2}-\frac{\alpha_i(T)}{2\beta_i(T)},\frac{1-\alpha_i(T)}{\beta_i(T)}\right),0\right].$$

It can be shown from the formulation of $z_i^h(t)$ that the heuristic has the properties that the offered discount increases with higher inventory and decreases with greater waiting demand. Further, as the number of waiting demands increase, the cut-off value $X_i^*|t,\Sigma\vec{d}$ increases. The base stock level for the TSH is given by solving the single stage problem assuming that demand for each class in the stage is distributed according to the demand for the entire period, i.e., as in $d_{i,1}(0,T]$.

The computational time of TSH is almost real-time, since the considered two-stage problem is simplified to a modified single stage problem. Because the values of $z_i^h(t)|X(t),\Sigma\vec{d}(t)$ and $X_i^*|t,\Sigma\vec{d}$ can be stored for bounded demand functions, the heuristic can in practice be applied in real-time.

Intuitively, the TSH should be very close to the optimal solution, since it can adjust the allocation and delivery decision solution according to the observed demand information. Observe that the optimal expected profit for a single stage problem in which a single decision is made after observing all of the demand is an upper bound of the expected profit for the TSH since the inventory allocation is globally optimal.

To illustrate the effectiveness of the TSH heuristic, we present computational results in Table 2.2. We experimented with the following data set: $c_p = 10$, $h = 8$, $K = 2$, $\alpha = (0,0)$ and $p = (p_1, 20)$, $\beta = (\beta_1, 0.01)$. The class demand $d_1(t)$ and $d_2(t)$, $t \in (0,100]$, follow a pair of independent Poisson distributions with arrival rates $\lambda_1 = 0.9$ and $\lambda_2 = 0.1$. We again varied the parameters p_1 and β_1. From Table 2.2, it can be seen that TSH was within less than 0.5% from the upper bound to the optimal ADP expected profit (upper bounding approach described in previous page with the use of the one stage problem). Further, the TSH solution resulted in total fill rates per class (i.e., total demand per class eventually met) that were within 2% of those provided by the optimal ADP solution, with the TSH solution provided higher fill rates for class-1 and lower for class-2 than it is in the profit optimizing solution. (The same nature and magnitude of performance holds over larger numerical sets we experimented with, but not reported here for brevity of presentation reasons).

Parameters		Expected Profit		Total Fill Rate Per Class			
p_1	β_1	ADP	TSH	ADP / $i = 2$	TSH / $i = 2$	ADP / $i = 1$	TSH / $i = 1$
12	5.0000	274.6168	273.5470	100%	99.07%	100%	100%
14	1.2500	446.4792	444.5983	100%	98.47%	100%	100%
16	0.3125	603.1823	601.1290	100%	98.11%	99.52%	99.66%
18	0.0781	758.8521	757.4440	100%	98.19%	96.99%	97.24%
20	0.0195	926.6229	926.1622	100%	98.08%	96.98%	97.19%

Table 2.2: Accuracy of Expected Profit and Total Fill Rate Per Class Performance.

2.5. A Comparison of the ADP Solution to a FCFS Policy

In this section we compare the optimal solution of the ADP to a First-Come/First-Serve (FCFS) policy. This discussion illustrates the role of dynamic inventory rationing and price discounting in the optimal solution.

2.5.1. First-Come-First-Serve (FCFS) Policy

The First-Come-First-Serve policy is a simplified ADP policy without inventory rationing consideration. We use the superscript f to identify the variables and functions under the FCFS policy throughout the paper. The profit for the two-stage problem is

$$\begin{aligned}\pi_j^f(z_j^f \mid X_j^f, d_j) = & \frac{\min\left(X_j^f, D_j\right)}{D_j} \sum_{i=1}^{K} p_i d_{ij} - c_j\left(\min\left(X_j^f, D_j\right) - g_j\right)^+ \\ & + \frac{\left(D_j - X_j^f\right)^+}{D_j} \sum_{i=1}^{K} d_{ij}\left[\gamma_{ij}^f\left(p_i - z_{ij}^f - c_p - c_w\right) - c_l\left(1-\gamma_{ij}^f\right)\right] \\ & + E_{d_{j+1}}\left\{\Pi_{j+1}^f\left[\left(X_j^f - D_j\right)^+, d_{j+1}\right]\right\}\end{aligned} \tag{2.17}$$

where $\Pi_{M+1}^f\left(X_{M+1}^f, d_{M+1}^f\right) \equiv \left(c_p - h\right)\left(X_M^f - D_M\right)^+$.

Observe that under this policy there exist no inventory rationing considerations, and we offer price discounts only if there is no inventory left. The problem the firm faces at stage j is

$$\Pi_j^f\left(X_j^f, d_j\right) \equiv \max_{0 \le z_{ij}^f \le \frac{1-\alpha_{ij}}{\beta_{ij}}} \pi_j^f(z_j^f \mid X_j^f, d_j) \tag{2.18}$$

The profit for the multiple stage FCFS problem is

$$\max_{X^f \ge 0} \Pi^f(X^f) \equiv -c_p X^f + E_{d_1}\left[\Pi_1^f\left(X^f, d_1\right)\right] \tag{2.19}$$

Proposition 2.12. *The unique optimal price discount at stage* j *of the FCFS problem is*

$$z_{ij}^{*f} = \max\left[\min\left(\frac{p_i + c_l - c_p - c_w}{2} - \frac{\alpha_{ij}}{2\beta_{ij}}, \frac{1-\alpha_{ij}}{\beta_{ij}}\right), 0\right] \tag{2.20}$$

Proof. Similar to Proposition 2.1.

Let $\gamma_{ij}^{*f} = \alpha_{ij} + \beta_{ij} z_{ij}^{*f}$. Observe that the discount z_{ij}^{*f} offered does not depend on X_j, d_j. This allows us to determine the optimal base stock independently below.

Proposition 2.13. *If* $E_{\sum_{k=1}^{j} D_k > X^f \geq \sum_{k=1}^{j-1} D_k}\left(\frac{d_{ij} L_{ij}^f}{D_j} - c_j 1_{\left(X^f - \sum_{k=1}^{j-1} D_k\right) \geq g_j}\right)$ *decreases in* j, *the optimal FCFS base stock* X^{*f} *is unique and solves*

$$\frac{\partial \Pi^f(X^f)}{\partial X^f}$$
$$= -c_p - h \cdot P\left(\sum_{j=1}^{M} D_j \leq X^f\right) + \sum_{j=1}^{M}\left[\begin{array}{l} P\left(\sum_{k=1}^{j} D_k > X^f \geq \sum_{k=1}^{j-1} D_k\right) \\ \cdot E_{\sum_{k=1}^{j} D_k > X^f \geq \sum_{k=1}^{j-1} D_k}\left(\sum_{i=1}^{K} \frac{d_{ij} L_{ij}^f}{D_j} - c_j 1_{\left(X^f - \sum_{k=1}^{j-1} D_k\right) \geq g_j}\right)\end{array}\right]$$
$$= 0$$

where $L_{ij}^f = p_i - \gamma_{ij}^{f*}\left(p_i - z_{ij}^{f*} - c_p - c_w\right) - \left(1 - \gamma_{ij}^{f*}\right) c_l$

Proof. The results follow from considering the first order conditions for the problem in (2.19). ■

2.5.2 Role of Dynamic Priorities, Inventory Rationing, and Price Discounts

We compare the expected profits, base stocks (X) and demand waiting rates (γ) for the FCFS solution to those of the optimal solution to the ADP problem. Because the FCFS solution is a feasible solution to ADP, its expected profits may be at most as large as those of the optimal solution. We observe that the optimal solution is achieved by both allocating inventory to the most profitable customers through their prioritization and by offering discounts to retain others. In the FCFS solution, only the latter, i.e. discounting, is retained.

As the following example illustrates, the comparison of base stock levels and demand waiting rates between FCFS and optimal ADP solution is non-conclusive. The optimal ADP solution may either increase or decrease, vis-à-vis the FCFS solution, the base stock levels and the demand waiting rates.

Example 2.1: Assume that $M=1$, $K=2$, and customer class demands are independent normal random variables $d_1, d_2 \sim N(\mu,\sigma)$. We have $D=d_1+d_2 \sim N\left(2\mu,\sqrt{2}\sigma\right)$, $E_{D>X}\left(d_i/D\right)=1/2$ and , from Proposition 4 and 13,

$$
\begin{aligned}
&E\left[\frac{\Delta\Pi^f(X,d)}{\Delta X}\right]-E\left[\frac{\Delta\Pi(X,d)}{\Delta X}\right] \\
&=P(D>X)\sum_{i=1}^{K}L_i^f E_{D>X}\left(\frac{d_i}{D}\right)-\sum_{i=1}^{K}L_i P(D_{i-1}\geq X>D_i) \\
&=\left(L_1\left[\frac{P(D>X)}{2}-P(D\geq X>d_2)\right]+L_2\left[\frac{P(D>X)}{2}-P(d_2\geq X)\right]\right) \\
&=\left(L_2-L_1\right)\left[\frac{P(D>X)}{2}-P(d_2\geq X)\right]
\end{aligned}
$$

Thus, $X^{f*}\geq X^*$ if and only if $P\left(D>X^{*f}\right)\geq 2P\left(d_2\geq X^{*f}\right)$.

For the case with $X^{f*}\geq X^*$, if $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$, we have $\gamma_2^*=\gamma_2^{*f}\geq\gamma_1^{*f}=\gamma_1^*$ from Proposition 2.1 and 2.12. The demand waiting rate under FCFS may be higher in certain cases than under the ADP policy. ■

We now look at the ADP and FCFS solutions without price discounts, i.e. given $z_{ij}=0$, or equivalently with $\beta_{ij}=0$. We have directly from Theorem 1:

Proposition 2.14: *For the multiple-stage ADP problem without price discount, the optimal priority sequence is to serve the customers in d_j in class order, from K to 1, prior to serving the waiting demand, $\vec{d}_j$. The optimal allocation policy is unique.* ■

Proposition 2.15. *If* $E_{\sum_{k=1}^{j} D_k > X^{f'} \geq \sum_{k=1}^{j-1} D_k} \begin{pmatrix} (d_{ij}/D_j)\left[p_i - \alpha_{ij}(p_i - c_p - c_w) - (1-\alpha_{ij})c_l\right] \\ -c_j 1_{(X^{f'} - D_{j-1}) \geq g_i} \end{pmatrix}$ *decreases on* j*, the optimal base stock level for the FCFS problem without price discount is unique.* ■

We compare the solutions with and without price discount for the ADP and FCFS solution to illustrate the role of price discounts. Obviously, the dynamic price discounts will increase the expected profit for either the optimal ADP or FCFS solution, as they wouldn't be used otherwise.

Proposition 2.16. *Use of price discounting reduces the profit maximizing base stock level for the FCFS policy.*

Proof. Given an inventory allocation X_j, the value of $E_{\bar{d}_j}\left[\partial \Pi_j / \partial X_j\right]$ with $z_{ij}^f = z_{ij}^{*f}$ is no larger than with $z_{ij}^f = 0$, since the demand denial penalties are minimized when $z_{ij}^f = z_{ij}^{*f}$. Thus, price discounting reduces the optimal base stock level, which satisfies $E_{\bar{d}_j}\left[\partial \Pi_j / \partial X_j\right] = 0$. ■

The result in Proposition 2.16 does not apply to the general ADP problem.

2.5.3. Numerical Insights

Example 2.2: Assume that $c_p = 1$, $h = 0.6$, $K = 2$, $p = (1.2, 2)$, $\alpha_j = (0,0)$, $\beta_j = (10, 0.1)$ for $j = 1,2$. All customer class demands d_{ij}, $i = 1,2$ and $j = 1,2$, are independent with each other and follow the same normal distribution with $\mu = 300$ and $\sigma = 90$. The congestion cost in the first stage is $c_1 = 0.6$ with $g_1 = 0$ and is zero in the second stage, i.e. $c_2 = g_2 = 0$. We also assume $c_w = c_l = 0.0$. (Note that non-zero c_l are effectively captured as part of the price parameter variations, and the price discount has a similar effect as c_w.)

Item name	ADP	ADP-Z=0	FCFS	FCFS-Z=0
Expected Profit	489.1698	427.7304	266.9272	262.2025
Base stock	932.0068	812.9883	1181.9	1192.3
Discount Cost	17.0220	0	2.7270	0
Total Fill Rate				
Average	99.98%	65.80%	97.08%	94.46%
Class-2	99.92%	99.46%	94.16%	94.46%
Class-1	100%	32.16%	100%	94.46%
Prompt Fill Rate				
Average	71.59%	65.80%	93.94%	94.46%
Class-2	99.92	99.46%	93.94%	94.46%
Class-1	43.26%	32.16%	93.94%	94.46%

Table 2.3: Results of Example 2.2

The ADP solution not only has the highest profit, but also the highest total fill rates. Inventory rationing increases the profit by 83.26%, reduces the base stock level by 21.14%, and increases the total fill rate of class-2 demand by 6.12%. The average total fill rate is also increased, even though the base stock is reduced.

The effect of price discounts for the case with inventory rationing is much more significant than without inventory rationing. For the case with inventory rationing, price discounting increases the expected profit by 14.36%, increases the average total fill rate by 51.95% and the class-1 total fill rate by 210.95%. Also, note that price discounting increases the base stock for the case with inventory rationing, but has the reverse effect for FCFS policies. While price discounting improves class-2 total fill rate for inventory rationing policies, it decreases the corresponding fill rate for FCFS ones.

Figure 2.1 a) __ b) report the effects of demand variance on expected profits and base stock levels for our various policies on the data set of Example 2.2. All the expected profits might decrease with the demand variance as is intuitive. For the optimal solution to the ADP, the profits are quite stable (profit reduction in most examples less than 0.1%) since most of the class 2 customers are served promptly and the cost of denying immediate demand satisfaction demand to the class-1 customers is small. The expected profit of the FCFS solution decreases more drastically with an increase in variance, as would be expected. Similar to a newsvendor-like formulation and for a base stock level less than the mean demand (as is the case in this example), an increase in the demand variance reduces the base stock level. We observe that the base stock for the optimal solution does not necessarily decrease with an increase in variance. Because of customer class prioritization, an increase in the variance increases the “marginal contribution” from the class-2 customers (i.e. the product of the probability of an additional unit being

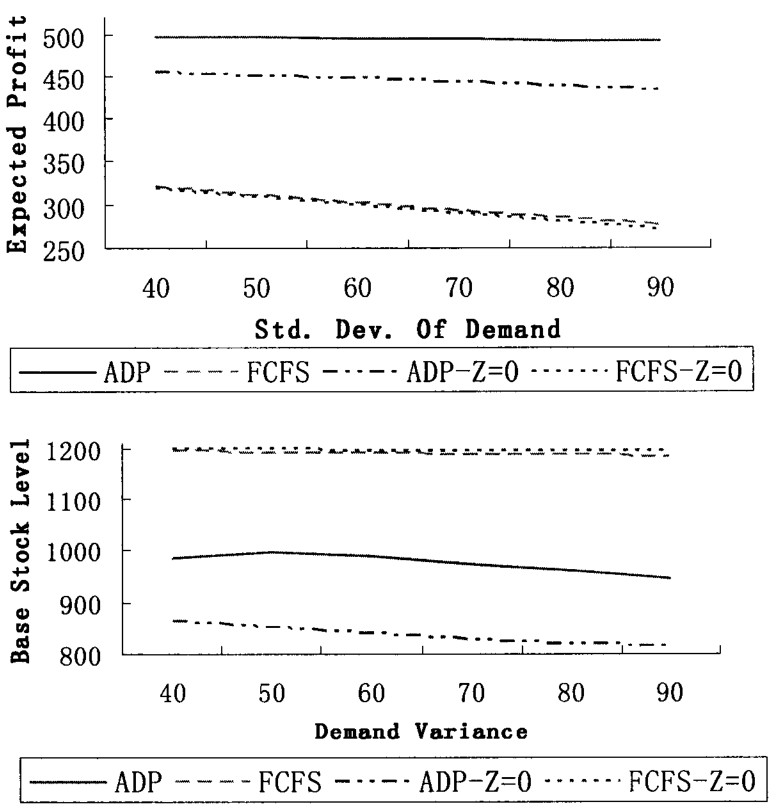
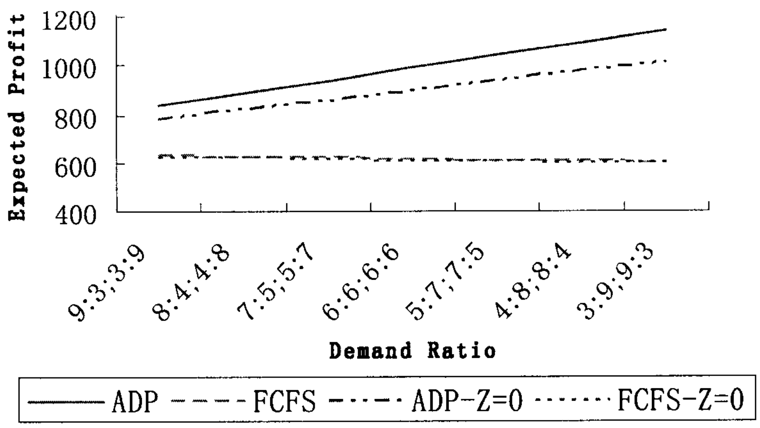


Figure 2.1 a)__b). Expected Profit and Base Stock Level vs. Demand Variance



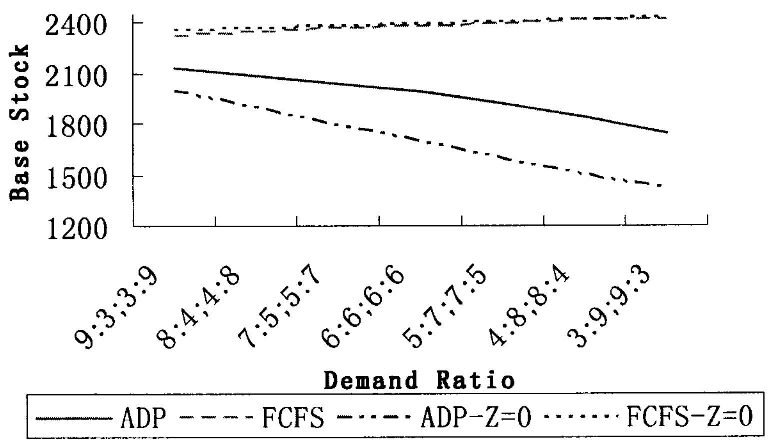


Figure 2.2 a)__b). Expected Profit and Base Stock Level vs. Demand Ratio

to class 2 customer times the profit margin of this customer), but decreases the "marginal contribution" from the class-1 customers. (This is in contrast to both customer class "marginal contributions" decreasing in the FCFS solution). The combined effect results in at first increasing and later decreasing with increasing variance of the base stock level.

In Figures 2.2 a) __ b) we change the ratio of the demand of the class-2 to class-1 customers in the first and second stages and present the expected profits and the base stocks. The ratios are reported $E(d_{2j})/E(d_{1j})$, for $j=1,2$, from $j=1$ to $j=2$, and show an increasing ratio of class-2 demand in the second stage reading the figures from left to right. (Note in these examples we let the mean total demand in each stage be 1200 and maintain a standard deviation of 90 for the demand from each class in each stage.) For the optimal solution to the ADP, the expected profit increases with the increasing demand 2 in the second stage because fewer units are allocated in the first stage where there is a congestion cost. The reduced congestion cost is greater than the increased cost of delaying the class-1 customers. The base stock decreases because more class-1 customers are denied demand in the first stage rather than incurring the congestion cost. However, for the FCFS policy, the expected profit decreases and the base stock increases with an increasing ratio of class-2 customers arriving in the second stage. More inventory is required to satisfy this later demand reducing the overall profitability.

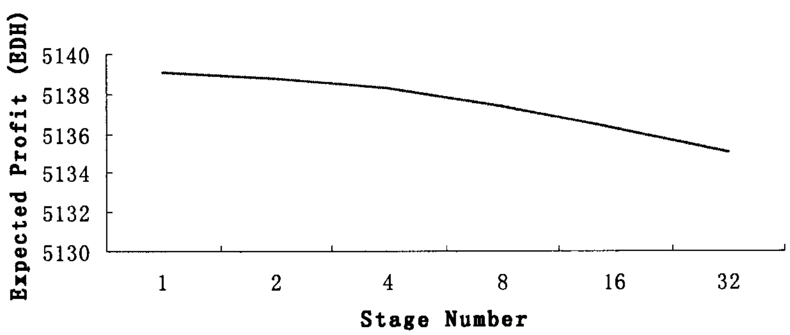
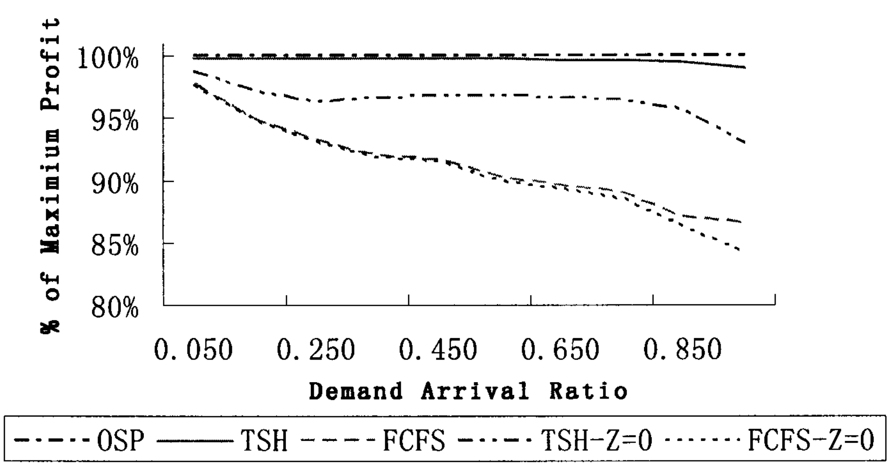


Figure 2.3. Expected Profit vs. Stage Number (Under EDH Policy)

In Figure 2.3 we report the expected profit preference of the EDH policy when the number of stages per period increases. We have implemented the following version of EDH policy. We are setting the starting base stock level according to a two stage EDH. At each stage the inventory allocation rule and price discounts are those of a two stage EDH with first stage demand the current stage demand and second stage demand the total demand of all remaining stages. As can be seen from the figure, the expected profit preference of the heuristic is robust with a slight, less than 0.1%, deterioration in expected profit performance with an increase in the number of stages. (For the example in Figure 2.3, the mean and variance of the demand are 2,000 and $\sqrt{2}\cdot(90)$ respectively, and there is no congestion cost). ■

Example 2.3: In this example, we compare the results of the two stage heuristic for the continuous time problem to the solution to the single stage problem and the FCFS solution. We assume that $c_p = 10$, $h = 8$, $c_w = c_l = 0.0$, $K = 2$, $p = (10.2, 20)$, $\alpha(t) = (0,0)$, $\beta(t) = (10.0, 0.01)$, $t \in (0,100]$. The class demand $d_1(t)$ and $d_2(t)$ follow a pair of independent Poisson distributions with arrival rates λ_1 and λ_2.

Figures 2.4 a) __ d) show the change in the expected profit as a percentage of the single stage problem upper bound, the base stock level and the fill rates as λ_1/λ_2, the ratio of class-1 to class-2 customers, increases. We observe that the TSH is effective in maintaining a high profit while there is a significant degradation in the profits when there is either no discounting ($z = 0$) or no inventory allocation as in the FCFS solution. The profit is maintained by appropriately setting the base stock level and then allocating inventory to the more profitable customers. The base stock for the TSH decreases since the class-2 customer's service is assured by inventory rationing. The base stock then levels off at an appropriate value when the marginal customer is most likely to be a class-1 customer. We observe that the class-2 prompt fill rate in the TSH is very high, with only some decline when they are in the vast majority or minority; there are either too many class-2 customers to serve all or too few to worry about. Of course this advantage comes at the price of class-1 prompt service.



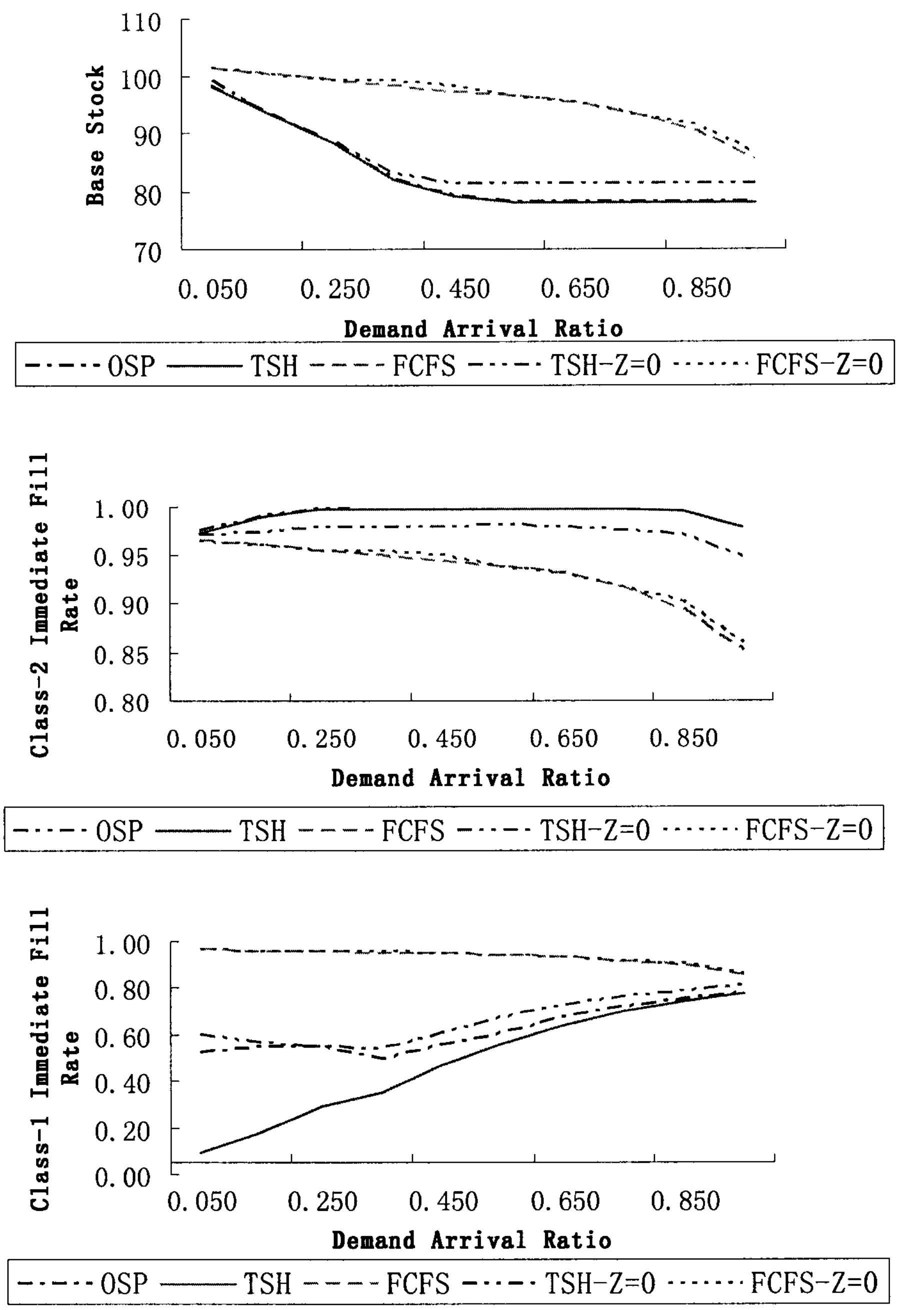


Figure 2.4. a) __ d). λ_1/λ_2 vs. a) Expected Profit as a Percentage of the Single Stage Problem Upper Bound; b) Base stock level; c) Class-2 Prompt Fill Rate; d) Class-1 Prompt Fill Rate

Figures 2.5. a) __ b) summarize nicely the desirable features of the ADP policies. The profit maximizing base stock level of such policies are lower than either the FCFS solution without discounting (effectively the newsvendor solution) or FCFS with dynamic discounting policies. In this example, stock level is reduced by more than 30%. Further the total fill rate is higher than any base stock level. At the profit maximizing level, the FCFS solution fill rate is more than 5% less than that of the ADP solution. Finally, the ADP solution provides increased profits, in this example more than 10% greater the newsvendor and FCFS policies.

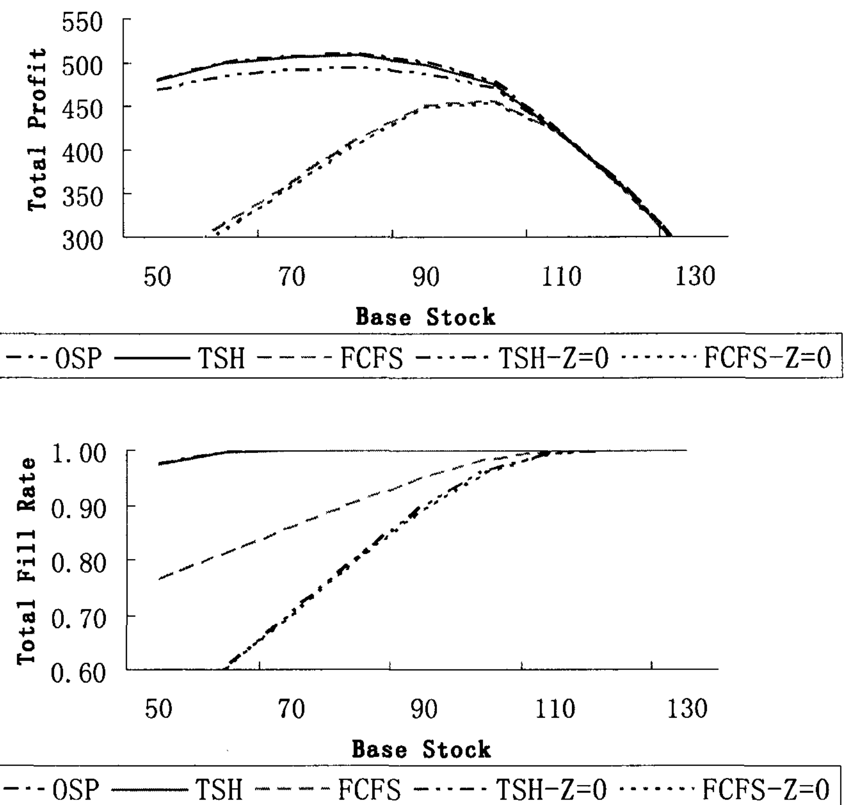


Figure 2.5. a) __ b). a) Total Profit; b) Total Fill Rate vs. Base Stock

Finally, in Table 2.4 we report results for an ADP problem with three customer classes. As apparent from the table, the TSH policy continues to exhibit impressively accurate performance in approximating both profits and fill rates of the ADP policy.

Class-Ratio			Base	Expected Profit		Total Fill Rate Per Class (1, 2 and 3)					
1	2	3	Stock	ADP	TSH	ADP	TSH	ADP	TSH	ADP	TSH
0.1	0.8	0.1	281.25	1514.3	1512.2	100%	100%	99.21%	99.24%	100%	99.67%
0.2	0.7	0.1	268.07	1407.9	1406.3	100%	100%	99.93%	99.89%	100%	99.94%
0.3	0.6	0.1	262.21	1287.6	1286.1	100%	100%	100%	99.95%	100%	99.94%
0.4	0.5	0.1	262.21	1171.4	1170.5	100%	100%	100%	99.96%	100%	99.96%
0.5	0.4	0.1	262.21	1045.3	1044.4	100%	100%	100%	99.96%	100%	99.94%
0.6	0.3	0.1	262.21	928.4	927.5	100%	100%	100%	99.95%	100%	99.91%

Table 2.4: Accuracy of Expected Profit and Total Fill Rate Class-Ratio Performance

2.6. Conclusions

In this paper, we are concerned with a problem of allocating inventory to demand from multi-class customers when partial backlogging of unfulfilled demand is possible. The probability of this occurring is influenced by dynamic discounts the firm may offer. We present a solution approach to the problem of determining the inventory allocation, the customer discounts and the prioritization of demand for all stages (referred to as ADP problem), through dynamic programming starting first with the final stage and then solving the problem by induction. We also consider the continuous-time demand case and provide an efficient and robust heuristic for its solution in real-time.

In our numerical analysis we study the role of inventory rationing and price discounting on the ADP solution while varying several parameters including the demand mean and variance, and the ratio of class-1 to class-2 customers. The ADP solution always increases the expected profit vis-à-vis the FCFS or no price discounting solutions. By increasing the demand waiting rate, ADP policies reduce the base stock level as well as the incurred holding and congestion costs in almost all cases.

On a technical aspect, the ADP solution is much more complex than its counterpart for the complete lost-demand case. Because waiting demand affects the performance of the later stages, the decisions among the stages are dependent on each other. Under assumptions of time homogeneity, we are able to show that the marginal value of inventory decreases in time, the allocation Y_j^* increases in time, the price discounts z_{ij}^* decreases in time. Further we show that the allocation in earlier periods decreases if the number of high-class customer arriving later increases compared with lower-class customers.

As potential research extensions, one can consider more general contexts to the ADP problem. For example, one could consider class-dependent lost demand penalties or class-dependent constraints on the achieved fill rate. On the other hand, the current ADP problem context allows for minor variations of certain assumption without any significant effects to its solution. For example, alternative interpretation of the congestion cost as stage holding cost, or time-dependent demand delaying penalty, leads to similar results. The ADP approach works well for multiple-units customer demands, if the total demand is much larger than the size of any one customer order. In our future research, we intend to study the design of profit maximizing contracts exploiting differences in prices and service levels for different customer classes when the firm intends to use ADP policies in serving those customers.

2.7 References

P. P. Belobaba, 1987. Airline Yield Management: An overview of seat inventory control. *Transportation Science*. 21, 63-73.

G. R. Bitran and S. V. Mondschein, 1997. Periodic Pricing of Seasonal Products in Retailing. *Management Science*. 43, 64-79.

S. L. Brumelle and J. I. McGill, 1993. Airline Seat Allocation with Multiple Nested Fare Classes. *Operations Research*. 41, 127-137.

K. D. Cattani and G. C. Souza, 2002. Inventory Rationing and Shipment Flexibility Alternatives for Direct Market Firms. *Production and Operations Management*. 11, 4, 441-457.

L. M. Chan, D. Simchi-Levi and J. Swann, 2001. Dynamic Pricing Strategies. *Working Paper*, University of Toronto.

F. Chen, 2001. Market Segmentation, Advanced Demand Information, and Supply Chain Performance. *Manufacturing & Service Operations Management*. 3, 1, 53-67.

K. L. Cheung, 1998. A continuous review inventory model with a time discount. *IIE Transactions*, 30, 747-757.

M. A. Cohen, P. R. Kleindorfer, and H. L. Lee, 1988. Service Constrained (s,S) Inventory Systems with Priority Demand Classes and Lost Sales. *Management Science*. 34, 482-499.

J. D. Dana, Jr., 1999. Using Yield Management to Shift Demand when the Peak Time is Unknown. *Rand Journal of Economics*. 30, 3, 456-474.

G. A. DeCroix and A. Arreola-Risa, 1998. On offering economic incentives to backorder. *IIE Transactions*. 30, 715-721.

G. V. Deshpande, M. A. Cohen and K. Donohue, 2003. A Threshold Inventory Rationing Policy for Service-Differentiated Demand Classes. *Management Science*. 49, 6, 683-703.

F. de Vericourt, F. Karaesmen, and Y. Dallery, 2001. Assessing the benefits of Different Stock-Allocation Polices for a Make-to-Stock Production System. *Manufacturing & Service Operations Management*. 3, 1, 53-67.

A. Federgruen and A. Heching, 1999. Combined Pricing and Inventory Control Under Uncertainty. *Operations Research*. 47, 454-475.

Y. Feng and B. Xiao, 2000. Optimal Policies of Yield Management with Multiple Predetermined Prices. *Operations Research*. 48, 332-343.

K. C. Frank, R. Q. Zhang and I. Duenyas, 2003. Optimal Policies for Inventory Systems with Priority Demand Classes. *Operations Research*. 51, 6, 993-1002.

I. L. Gale and T. J. Holmes, 1993. Advance-Purchase Discounts and Monopoly Allocation of Capacity. *American Economic Review*. 83, 1,135-146.

G. Gallego and G. J. van Ryzin, 1994. Optimal Dynamic Pricing of Inventories with Stochastic demand Over Finite Horizon. *Management Science*. 40, 999-1020.

Y. Gerchak, M. Parlar and T. K. M. Yee, 1985. Optimal Rationing Policies and Production Quantities for Products with Several Demand Classes. *Canadian Journal of Administrative Science*. 2, 161-176.

A. Ha, 1997. Inventory Rationing in a Make-to-Stock Production System with Several Demand Classes and Lost Sales. *Management Science*. 43, 1093-1103.

J. I. McGill and G. J. van Ryzin, 1999. Revenue Management: Research Overview and Prospects. *Transportation Science*. 33, 233-256.

L. W. Robinson, 1995. Optimal and Approximate Control Policies for Airline Booking with Sequential Nonmonotonic Fare Classes. *Operations Research*. 43, 252-263.

G. T. Thowsen, 1975. A Dynamic, Nonstationary Inventory Problem for a Price/Quantity Setting Firm. *Naval Research Logistics*. 22, 461-476.

D. M. Topkis, 1968. Optimal Ordering and Rationing Policies in a Nonstationary Dynamic Inventory Model with n Demand Classes. *Management Science*. 15, 160-176.

L. R. Weatherford and S. Bodily and Pfeiffer, 1993. Modeling the Customer Arrival Process and Comparing Decision Rules in Perishable Asset Revenue Management Situations. *Transportation Science*. 27, 239-251.

Y. Wang, M. A. Cohen and Y.-S. Zheng, 2002. Differentiating Customer Service on the Basis of Delivery Lead Times. *IIE Transactions*. 34, 11, 979-989.

R. D. Wollmer, 1992. An Airline Seat Management Model for a Single Leg Route when Lower Fare Classes Book First. *Operations Research*. 40, 26-37.

E. Zabel, 1972. Multi-period Monopoly under Uncertainty. *Journal of Economic Theory*. 5, 524-536.

W. Zhao and Y. S. Zheng, 2000. Optimal Dynamic Pricing of Perishable Assets with Nonhomogeneous Demand. *Management Science*. 46, 375-388.

Chapter 3

Optimizing Multi-Class Deterministic Demand Fulfillment through Dynamic Pricing and Inventory Rationing

3.1 Introduction

3.1.1 Problem Motivation

In this paper we study the deterministic demand inventory problem of allocating inventory and offering price discounts to several classes of customers when partial backlogging of unfilled demand is possible. The customer classes are distinguished by the price they are to pay for the item and their willingness to wait for fulfillment of demand with a reasonable delivery delay. The firm may offer a customer either a prompt service or a delaying delivery service with the associated price discounts according to the current inventory information. The probability of customer waiting rate is influenced by some class specific parameters. The inventory is replenished by a continuous review policy. We focus on the deterministic demand problems in this paper, including the problem with constant arriving rates on the infinite horizon, or called the Economic Order Quantity (EOQ) problem, and the time varying demand problem, or called the Dynamic Economic Lotsize (DEL) problem, and the problem with multiple stages.

The multi-customer-class problem occurs in many practical environments. For examples, some customers may choose higher contractual price associated with higher service level. Some computer companies ask customers to choose warranty contracts contingent on the sold computers, or offer price discounts for delivery delay. The delivery cost of the firm, which can be treated as a kind of price in this paper, may be different among the customer classes, for the sake of order sizes or customer locations. Dekker Kleijn and De Rooij (1998) discuss a case studying on the inventory control of spare parts in a large petrochemical plant, where parts are installed in equipment of different criticality. Kleijn and Dekker (1998) list more real applications in their survey paper.

For the deterministic inventory problems with single customer class, Zipkin (2000) provides a context book including extensive results for the EOQ, DEL and multi-stage problems. On the other hand, the rationing and pricing results for the multi-class deterministic demand problems are rarely explored. As one of the results closest to our work, Moon and Kang (1998) provide an optimal rationing level in a continuous review

(Q, r) policy for an EOQ model with two customer classes and the unfulfilled demands completely backordered, while the value of Q and r are given constants.

In this paper, we introduce a frame work for the multi-class demand problem with inventory rationing and dynamic price discounting in the deterministic and partial backlogging context. We use the prompt service welfare to measure the firm's benefit if offering prompt service to a customer. This concept not only reduces the technical burden by calculating the inventory cost associated to the customer at his arriving time, but also interprets the firm's incentives of allocating inventory and offering price discounts. We propose an optimal continuous review policy for the EOQ model (still called EOQ policy), and illustrate that both inventory rationing and price discounting can increase the average profit and customer fill rates significantly. Then, we apply the generalized EOQ policies for some time varying demand and multi-stage problems, as well as explore the efficient heuristics for the considered frame work.

3.1.2 Literature Review

Our research is also related to work in inventory rationing, offering economic incentives for customer retention and revenue management. Early work by Topkis (1968) considers the rationing of inventory to demand from n customer classes when a period is divided into several intervals. He shows that a base-stock ordering policy is optimal and demand is fulfilled in class order as long as inventory is above a class-dependent allocation level. A model similar to Topkis with two priority demand classes, one deterministic and the other stochastic, has been considered by Frank, Zhang and Duenyas (2003). Nahmias and Demmy (1981) were the first to analyze a rationing policy in a continuous review environment. They consider a two-customer-class system with Poisson demand. They focus on evaluating fill rates for given rationing and reorder policy. Moon and Kang (1998) generalized their results to the problem with n class demands. Melchiors, Dekker and Kleijn (1998) consider a similar problem within a lost sales context. Deshpande, Cohen and Donohue (2003) analyze a static threshold-based rationing policy with optimal parameters for a similar model with two demand classes. Cohen, Kleindorfer and Lee (1988) consider an *(s, S)* inventory system where two classes of customers arrive, with the higher priority customer being served first. Their paper focuses on determining the reorder level s and the order-up-to value S through the development of heuristics and approximations. A review on rationing literature is provided by Kleijn and Dekker (1998). In a recent paper, Cattani and Souza (2002) investigate inventory-rationing policies in multi-class fixed price environments, where prices have been a priori set, with application for firms operating in a direct market channel. They compare the performance of these rationing policies with a pure first-come, first-serve policy under various scenarios for customer response to delay (lost sales, backlog, and a combination of lost sales and backlog). The inventory system is fed by a production system or co-located supplier with exponentially distributed processing times.

Most rationing literature considers either the completely lost sales context or the completely backorder context. It is not necessary to offer price discounts for these two special cases. Besides, as far as we know, there is no optimal policy provided to the continuous-review inventory rationing problem yet, though the heuristics have been studied for more than twenty years. In this chapter, we introduce an optimal dynamic economic order policy associated with inventory rationing and dynamic price discounting in the partial lost sales context.

Another related stream of work in the inventory management literature looks on economic incentives to retain customers in the presence of stockouts. Cheung (1998) considers a continuous review model where a discount can be offered to the customers willing to accept backorders or substitutable units even before the inventory is depleted, but the proportion of backordering customers is not a function of monetary incentives, as is the case in our work. Gale and Holmes (1993) consider a similar model in the economic literature. They discuss the use of advance-purchase discounted prices to divert demand from a peak period flight to an off peak period flight. They show by doing so, a monopoly airline would expand output and total surplus. The optimality of offering backordering incentives for a simple inventory system is explored in DeCroix and Arreola Risa (1998), but their analysis does not exploit different customer classes or dynamic discount adjustments.

Chen (2001) inspired by e-retailing environments studies optimal pricing-replenishment strategies that balance the costs due to discounted prices and the benefits due to advance demand information from customers willing to accept longer lead times for the right discount. In the paper the firm offers a menu of price and lead-time combinations, and customers can choose their priorities. Chen focuses on finding an optimal menu of static prices, as opposed to dynamic discounts we use, and he does not consider inventory-rationing policies, which are essential elements of our formulation. Wang, Cohen and Zheng (2002) study a problem of meeting demand from two demand classes with different lead time requirements. However, their paper focuses on studying required inventory levels in a two echelon supply chain, where each location follows a base stock policy with no inventory rationing, whereas we specifically focus on dynamic pricing and inventory rationing for a single location.

For the research work related to the integrated consideration of inventory rationing and dynamic pricing, Ding, Kouvelis and Milner (2003) consider a periodic inventory problem with multiple classes of customers. They show how to determine the inventory allocation, dynamic discounting and customer-class prioritization in a model where each period is divided into a number of stages. They also provide an efficient and robust heuristic for the continuous time demand case. Different from optimizing the profit in a periodic review model, this paper considers the continuous review inventory problem with multi-class deterministic demands.

3.1.3 Paper Organization

We organize this paper as follows: Introduce the infinite horizon problem in Section 3.2. Determine the EOQ policy in Section 3.3. Illustrate the role of inventory rationing and price discounting in Section 3.4. Explore the optimal and heuristic policies for the DEL and multi-stage problems in Section 3.5 and 3.6, respectively. Finally, survey the conclusions in Section 3.7.

3.2 Formulation of the Infinite Horizon Problem

Based on the problem description, we consider a model in which there are N customer classes and let subscript $i=1,\cdots,N$ denote the customer class. Let the per unit revenue from class i be p_i and, without loss of generality, assume $p_1 \le \cdots \le p_N$. Assume the demand of class i arrives at a constant rate λ_i in an infinite horizon problem. There is no lead time or capacity limit on an order.

We consider a continuous review policy in which the demand fulfillment is rationed based on customer classes, i.e. we may deny prompt service for some low profit customers and hold the inventory for some high profit customers when the on-hand inventory level is lower than a certain level. Assuming the system is in the partial lost sales context, we can offer a price discount to increase the customer retention when the prompt service is denied. An order is triggered at an reorder point R with an order-up-to base stock level Q, and an order policy is defined as (Q,R). (We will show that R is determined by the total per-time-unit backorder cost, not by the number of backorders at the reorder point.)

Given an order policy (Q,R), when a class i customer arrives at time t, the firm may offer one of two types of services: the prompt service and a delaying service with an associated per-time-unit price discount z_i. We describe a service package as (y_i,z_i), where $y_i=1$ means that the prompt service is available; otherwise $y_i=0$. Let (Y,Z) denote the set of (y_i,z_i), where Y and Z represent the rationing policy and price discounting policy, respectively.

Let the probability that a customer will wait be γ_i for the delaying service and we assume for simplicity that it is defined by the linear function

$$\gamma_i = \alpha_i + \beta_i z_i$$

That is, those customers choose to wait based on independent Bernoulli random variables. We restrict $0 \le z_i \le (1-\alpha_i)/\beta_i$ so that $0 \le \gamma_i \le 1$.

Finally, we introduce some constant parameters:

K : Setup cost;

c : Per-unit production cost; assume $p_i - c \geq 0$;

h : Per-time-unit holding cost;

a_i : Per-time-unit lost demand penalty for class i;

b_i : Per-time-unit backorder cost for class i;

The firm's objective is to determine a decision policy (Y,Z,Q,R) to maximize its average profit. Since there is no capacity limit, an order can clean all backorders and set up the on-hand inventory at an ideal *base-stock level*, denoted as Q. Q is identical to every order, since we consider an infinite horizon problem and the demand structure after each order is the same. The time interval between two adjoining orders is defined as a *cycle*. The cycle time, denoted as $T(Y,Z,Q,R)$, is unique, since the demand rate is constant. Maximizing the objective function is equivalent to maximizing the average profit in such a cycle.

Let $(Y,Z,Q|T)$ denote a decision policy conditional on given T, and $\Pi(Y,Z,Q|T)$ denote the sum of selling revenue, purchasing cost, holding cost and backorder cost in the cycle. Let $(Y^*,Z^*,Q^*|T)$ denote an optimal cycle policy. The *cycle* problem is defined as

$$\Pi(Y^*,Z^*,Q^*|T) = \max_{Y,Z,Q}\left[\Pi(Y,Z,Q|T)\right] \tag{3.1-a}$$

Then, we find an optimal cycle time T^* to maximize the average profit among all cycle times

$$\frac{\Pi(Y^*,Z^*,Q^*|T^*)-K}{T^*} = \max_{T}\left[\frac{\Pi(Y^*,Z^*,Q^*|T)-K}{T}\right] \tag{3.1-b}$$

Let $R(Y,Z,Q|T)$ denote the per-time unit backlogging cost at the end of cycle. It is easy to see that a decision policy is optimal in (3.1-b) if and only if it is also optimal in the counterpart problem defined by the traditional approach as following

$$\frac{\Pi(Y^*,Z^*,Q^*,R^*)-K}{T(Y^*,Z^*,Q^*,R^*)} = \max_{Y,Z,Q,R}\left[\frac{\Pi(Y,Z,Q,R)-K}{T(Y,Z,Q,R)}\right]$$

where the reorder point is uniquely corresponded to the cycle time by $R\left(Y,Z,Q|T\right)$.

This problem involves both rationing and discounting, called the "RD" problem.

3.3 Solution of the RD Problem

Given T and applied a policy $(Y,Z,Q|T)$ in a cycle, the profit function for individual customer can be constructed as follows. When a class i demand arrives at $t \in (0,T]$, given discount rate z_{it} if $y_{it}=0$, there are three cases for the associated holding, backlogging and lost demand cost: 1) ht, if the customer is served promptly; 2) a_i, if the demand is lost; 3) $(z_{it}+b_i)(T-t)$, if the customer is served by a unit from the next order. The customer's profit function is defined as

$$\pi_i(t, y_{it}, z_{it}|T) = \begin{cases} p_i - c - ht & y_{it} = 1 \\ \gamma_i\left[p_i - c - (z_{it}+b_i)(T-t)\right] - (1-\gamma_i)a_i & y_{it} = 0 \end{cases} \quad (3.2)$$

The total profit from customers in $(t,T]$ is $\sum_{i=1}^{N}\lambda_i\int_t^T \pi_i(\tau, y_{i\tau}, z_{i\tau}|T)d\tau$. The profit function for the RD problem is

$$\Pi(Y,Z,Q|T) = \sum_{i=1}^{N}\lambda_i\int_0^T \pi_i(\tau, y_{i\tau}, z_{i\tau}|T)d\tau \qquad \text{s. t.} \quad Q = \sum_{i=1}^{N}\lambda_i\int_0^T y_{i\tau}d\tau \quad (3.3)$$

The following proposition determines the optimal price discounts given an order policy, allocation policy and a cycle time.

Proposition 3.1. *Given $(Y,Q|T)$, the optimal price discount and waiting rate increase in time and are uniquely determined as*

$$z_{it}^* = \max\left[\min\left(\frac{p_i+a_i-c}{2(T-t)} - \frac{b_i}{2} - \frac{\alpha_i}{2\beta_i}, \frac{1-\alpha_i}{\beta_i}\right), 0\right] \text{ and } \gamma_{it}^* = \alpha_i + \beta_i z_{it}^*, \quad (3.4)$$

Comments Observe that the offered discount is only dependent on the remaindered cycle time and the customers' own parameters. Since the cycle time is fixed in the cycle problem (1-a), the optimal price discount is independent with the order policy and allocation policy. This allows us to determine the decision variables Y and Q easily in the later discussions.

Proof. Given $(Y,Q|T)$, when a class i customer arriving at $t\in(0,T]$, we add an incremental price discount Δz_{it}. Δz_{it} only affects the profit from the customer *it* if $y_{it}=0$ and $0\le\gamma_{it}<1$. The incremental profit is

$$\frac{\Delta\Pi}{\Delta z_{it}}=\frac{\Delta\pi_i(t,0,z_{it}|T)}{\Delta z_{it}}=\beta_i\left[p_i+a_i-c-(z_{it}+b_i)(T-t)\right]-(\alpha_i+\beta_i z_{it})(T-t).$$

This results (4) from the first order condition, noting the constraint condition that the price discount is non-negative and the waiting rate is less than one. ■

Let $w_i(t,z_{it}|T)$ be the incremental profit by switching a delaying service with discount z_{it} to the prompt service when a class i customer arrives at t, called the *prompt service welfare*. We obtain

$$\begin{aligned}&w_i(t,z_{it}|T)=\pi_i(t,1|T)-\pi_i(t,0,z_{it}|T)\\&=(p_i-c-ht)-\left(\gamma_i\left[p_i-c-(z_{it}+b_i)(T-t)\right]-(1-\gamma_i)a_i\right)\\&=(1-\gamma_i)(p_i+a_i-c)+\gamma_i(z_{it}+b_i)(T-t)-ht\end{aligned}\tag{3.5}$$

We will show later in Proposition 3.2 that $w_i(t,z_{it}^*|T)$ is decreasing in $t\in[0,T]$. There exists at most one solution, denoted as t_i^*, satisfying $w_i\left(t,z_{it_i^*}^*|T\right)=0$, called the *allocation threshold time* of class i demand and Let $t_i^*=T$ if $w_i\left(t,z_{it_i^*}^*|T\right)\ge 0$, for all $t\in[0,T]$.

Proposition 3.2. *Given T and applied Z^*, the allocation threshold time t_i^* is uniquely determined as*

$$t_i^*=\min\left[\frac{\left(1-\gamma_{it_i^*}^*\right)(p_i+a_i-c)+\gamma_{it_i^*}^*\left(z_{it_i^*}^*+b_i\right)T}{h+\gamma_{it_i^*}^*\left(z_{it_i^*}^*+b_i\right)},T\right].\tag{3.6}$$

If $0\le\alpha_N\le\cdots\le\alpha_1\le 1$ and $0\le\beta_N\le\cdots\le\beta_1$, then $0\le t_1^*\le\cdots\le t_N^*\le T$.

The optimal base stock level is uniquely determined as

$$Q^* = \sum_{i=1}^{N} \lambda_i t_i^* . \tag{3.7}$$

The total per-time-unit backorder cost at the associated reorder point is uniquely determined as

$$R^* = \sum_{i=1}^{N} \lambda_i \int_{t_i^*}^{T} \gamma_{i\tau}^* \left(z_{i\tau}^* + b_i\right) d\tau \text{ and } r^* = r\left(Y^*, Z^*, Q^* \middle| T\right) = \sum_{i=1}^{N} \lambda_i \int_{t_i^*}^{T} \gamma_{i\tau}^* d\tau \tag{3.8}$$

where r^* is the number of backlogged demand at the end of cycle.

The optimal rationing rule Y^* is to allocate the units to the demand arriving at $t \le t_i^*$.

Comments The concept of prompt service welfare focuses on the profit of individual customer during the whole cycle time. This intuitive approach determines the allocation threshold time of each customer class independently given the cycle time. Consequently, the order policy follows directly.

Proof. Applying the optimal price discount policy Z^*, we add an incremental unit on Q and allocating it to a class i demand arriving at time t, which is denoted as Δy_{it}. The incremental cycle profit, denoted as $\Delta\Pi/\Delta y_{it}$, is equal to the benefit from this customer and

$$\Delta\Pi/\Delta y_{it} = w_i\left(t, z_{it}^* \middle| T\right) = \left(1 - \gamma_{it}^*\right)\left(p_i + a_i - c\right) + \gamma_{it}^*\left(z_{it}^* + b_i\right)\left(T - t\right) - ht .$$

Let u and v be two times such that $0 < u \le v < T$. Since z_{iv}^* minimizes $w_i\left(t, z_{iv}^* \middle| T\right)$, choosing $z_{iv} = z_{iu}^*$, we have $\Delta\Pi/\Delta y_{iv} \le \Delta\Pi/\Delta y_{iv}\big|_{z_{iv}=z_{iu}^*}$ and

$$\Delta\Pi/\Delta y_{iu} - \Delta\Pi/\Delta y_{iv} \ge \Delta\Pi/\Delta y_{iu} - \Delta\Pi/\Delta y_{iv} \big| z_{iv} = z_{iu}^*$$
$$= \frac{1}{T}\left[h + \gamma_{iu}^*\left(z_{iu}^* + c_{bi}\right)\right]\left(v - u\right) \ge 0$$

which implies that the marginal profit decreases with time. There exists either a unique solution t_i^* for $w_i\left(t, z_{iv}^* \middle| T\right) = 0$, or $t_i^* = T$ if $w_i\left(T, z_{iT}^* \middle| T\right) > 0$ for all $t \in [0, T]$, and results (3.6).

We compare t_x^* and t_y^*, where $x > y$ are two class indexes. Choosing $\gamma_{yt} = \gamma_{xt}^*$, we have $\Delta\Pi/\Delta y_{yt} \le \Delta\Pi/\Delta y_{yt}\big|_{\gamma_{yt}=\gamma_{xt}^*}$ from γ_{yt}^* minimizing $\Delta\Pi/\Delta y_{yt}$, and $z_{xt}^* \ge z_{yt}\big|_{\gamma_{yt}=\gamma_{xt}^*}$ from $\alpha_x \le \alpha_y$ and $\beta_x \le \beta_y$. Thus we got

$$\begin{aligned} &\Delta\Pi/\Delta y_{xt} - \Delta\Pi/\Delta y_{yt} \ge \Delta\Pi/\Delta y_{xt} - \Delta\Pi/\Delta y_{yt}\big|_{\gamma_{yt}=\gamma_{xt}^*} \\ &= \left(1-\gamma_{xt}^*\right)\left(p_x + a_x - p_y - a_y\right) + \gamma_{xt}^*\left(z_{xt}^* + b_x - z_{yt}\big|\gamma_{xt}^* - b_y\right)(T-t) \ge 0 \end{aligned}$$

which implies $t_x^* \ge t_y^*$, since $\Delta\Pi/\Delta y_{yt}$ decreases with t.

The optimal solutions of $\left(Q^*, R^*, r^*\right)$ follow directly. ■

Proposition 3.3. *Given a policy* $\left(Y^*, Z^*, Q^* \middle| T\right)$, *the optimal cycle time* T^* *is uniquely determined by*

$$T(\partial\Pi/\partial T) - \Pi + K = 0 \tag{3.9}$$

where

$$\begin{aligned} &T(\partial\Pi/\partial T) - \Pi + K \\ &= T\left(\sum_{i=1}^{N} \lambda_i (p_i - c) - hQ^*\right) - \sum_{i=1}^{N} \lambda_i \int_0^T \pi_i\left(\tau, y_{i\tau}^*, z_{i\tau}^* \middle| T\right) d\tau + K \end{aligned} \tag{3.10}$$

decreases in T.

Proof. Given a policy $\left(Y^*, Z^*, Q^* \middle| T\right)$, we consider the first order condition in T. We get

$$\frac{\partial\left[(\Pi - K)/T\right]}{\partial T} = \frac{T(\partial\Pi/\partial T) - (\Pi - K)}{T^2} = 0$$

which is equivalent to (3.9).

Adding an increment ΔT on the cycle time, the term $\partial\Pi/\partial T$ can be calculated according to its discrete version.

$$\frac{\partial \Pi}{\partial T}=\frac{\Delta \Pi}{\Delta T}$$

$$=\frac{1}{\Delta T}\left[\sum_{i=1}^{N}\lambda_i\left[\int_0^{T+\Delta T}\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T+\Delta T\right)d\tau-\int_0^{T}\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T\right)d\tau\right]\right]$$

$$=\frac{1}{\Delta T}\left[\begin{array}{l}\sum_{i=1}^{N}\lambda_i\left(\int_0^{\min\left(t_i^*,t_i^*+\Delta t_i^*\right)}\left[\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T+\Delta T\right)-\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T\right)\right]d\tau\right)\\ +\sum_{i=1}^{N}\lambda_i\left(\int_{\max\left(t_i^*,t_i^*+\Delta t_i^*\right)}^{T}\left(\begin{array}{l}\pi_i\left(\tau,y_{i,(\tau+\Delta T)}^*,z_{i,(\tau+\Delta T)}^*\middle|T+\Delta T\right)\\-\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T\right)\end{array}\right)d\tau\right)\\ +\sum_{i=1}^{N}\lambda_i\left[\int_{\min\left(t_i^*,t_i^*+\Delta t_i^*\right)}^{\max\left(t_i^*,t_i^*+\Delta t_i^*\right)}\left[\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T+\Delta T\right)-\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T\right)\right]d\tau\right]\\ +\sum_{i=1}^{N}\lambda_i\left[\int_{\min\left(t_i^*,t_i^*+\Delta t_i^*\right)}^{\max\left(t_i^*,t_i^*+\Delta t_i^*\right)+\Delta T}\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T+\Delta T\right)d\tau\right]\end{array}\right] \quad (3.11\text{-a})$$

The first two terms in (3.11-a) are zero from $\pi_i\left(.\middle|T+\Delta T\right)=\pi_i\left(.\middle|T\right)$; the third term is zero from $w_i\left(t_i^*,z_{it_i^*}^*\middle|T\right)=0$ and the continuity of w_i and t_i^*; and the fourth term can be written as $\lambda_i\left(p_i-c-ht_i^*\right)/\Delta T$. Thus, we have

$$\Delta\Pi/\Delta T=\sum_{i=1}^{N}\lambda_i\left(p_i-c-ht_i^*\right)=\sum_{i=1}^{N}\lambda_i\left(p_i-c\right)-Q^* \quad (3.11\text{-b})$$

Since $\Delta\Pi/\Delta T$ decreases in Q^* from (3.11-b) and it is easy to see that Q^* increases in T, we have $\Delta\Pi/\Delta T$ decreases in T and $T\left(\partial\Pi/\partial T\right)-\Pi=\int_0^T\left(\partial\Pi/\partial T-\partial\Pi/\partial T(\tau)\right)d\tau$ decreases with T. Thus the optimal cycle time T^* in (3.9) is unique. The equation (3.10) holds and the decreasing argument follows from (3.11-b) and (3.3). ■

Intuitively, adding an increment ΔT on the cycle time, if a class i customer arrives at $t-\Delta T\in\left(-\Delta T,t_i^*-\Delta T\right]$, the profit from this customer is the same as that from a customer arriving at t when the cycle time is T. If $t_i^*=T$ and $w_i\left(T,z_{iT}^*\middle|T\right)\geq 0$, we add the marginal unit to the order and the marginal profit for class i customers is $\lambda_i\left(p_i-c-hT\right)$. If $t_i^*<T$ and a class i customer arrives at $t>t_i^*$, the profit from this customer is the same as that when the cycle time is T. The profit increment is the profit from customers arriving during $\left(t_i^* - \Delta T, t_i^*\right]$, where we have $w_i\left(t_i^*, z_{it_i^*}^* \middle| T\right) = 0$.

If $\beta_i > 0$, we have $z_{it}^* = (1-\alpha_i)/\beta_i$ and $w_i\left(t, z_{it}^* \middle| T\right) \to -ht < 0$ as $t \to T$. The allocation threshold times are inner solutions, i. e. $t_i^* < T$. Then, the result in Proposition 3.3 can also be driven by adding ΔT at the end of cycle. Note that the profit from the prompt service is unaffected. The profit from $\left(t_i^* + \Delta T, T + \Delta T\right]$ is the same as that from $\left(t_i^*, T\right]$ when the cycle interval is T. Thus, the incremental profit is the same as that in (11), since it is from the interval $\left(t_i^*, t_i^* + \Delta T\right]$ with $w_i\left(t_i^*, z_{it_i^*}^* \middle| T\right) = 0$.

If $\beta_i = 0$ and $T < (1-\alpha_i)(p_i + a_i - c)/h$, we have $w_i\left(t, z_{it}^* \middle| T\right) > 0$ for all $t \in [0, T]$ and offer all class i customer prompt service. This case implies that the stock-out is not allowed if the lost sales penalty is large than $a_i > c - p_i + hT/(1-\alpha_i)$. We will show more related results in Section 3.4.3 for the problem of no using price discounts.

We provide an upper bound of cycle time in the following algorithm by finding a solution such that $T(\partial\Pi/\partial T) - \Pi + K \le 0$. A cycle time satisfying $\partial\Pi/\partial T \le 0$ is an upper bound from $\Pi - K \ge 0$. An upper bound of allocation threshold time is then determined as $t^\wedge = (p_N - c)/h$ from $\partial\Pi/\partial T = \sum_{i=1}^N \lambda_i \left(p_i - c - ht^\wedge\right) \le 0$. Thus, we obtain an upper bound of cycle time from (3.6)

$$T^\wedge = \max_{1\le i\le N}\left\{\frac{\left(h + \gamma_{it^\wedge}^*\left(z_{it^\wedge}^* + b_i\right)\right)t^\wedge - \left(1-\gamma_{it^\wedge}^*\right)\left(p_i + a_i - c\right)}{\gamma_{it^\wedge}^*\left(z_{it^\wedge}^* + b_i\right)}\right\} \le t^\wedge\left(1 + \frac{h}{\min_{1\le i\le N}\left[\gamma_{it^\wedge}^*\left(z_{it^\wedge}^* + b_i\right)\right]}\right)$$

where $t^\wedge = (p_N - c)/h$.

Algorithm 3.1: Determine the RD Optimal Policy:

Initial Condition:

$T^< = 0$ and $T^> = T^\wedge$ are a pair of lower and upper bounds of cycle time.

Step 1. Choose the cycle time $T=\left(T^{<}+T^{>}\right)/2$, calculate z_{it}^{*}, t_{i}^{*} and Q^{*} directly according to (3.5), (3.6) and (3.7).

Step 2. Calculate Π^{*} from (3.3). If the condition in (3.9) and (3.10) holds, i.e., $T(\partial\Pi/\partial T)-\Pi+K=0$, then T is the optimal solution and go to Step 4.

Step 3. Set $T^{>}=T$ if $T(\partial\Pi/\partial T)-\Pi+K<0$; *Otherwise,* set $T^{<}=T$. Go to Step 1.

Step 4. Calculate R^{*} and r^{*} from (3.8). ■

In Algorithm 3.1, we need to search the optimal allocation threshold time and calculate the profit integral given a cycle time. Then search the optimal cycle time based on the monotony of the first order condition. The computational complexity is in $O\left(T^{\wedge}\log T^{\wedge}\right)$ time.

3.4 Role of Inventory Rationing and Price Discounting

3.4.1 First-Come-First-Serve (FCFS) Policy

In the FCFS policy, we offer prompt service as long as there are on-hand inventory. We offer price discounts only if there exists no inventory left. We use the superscript f to identify the variables and functions under the FCFS policy throughout the paper. Given a cycle time T^f, defining $\lambda = \sum_{i=1}^{N} \lambda_i$, the individual customer profit is similar to (3.2),

$$\pi_i^f\left(t, z_{it}^f, Q^f \middle| T^f\right) = \begin{cases} p_i - c - ht & t \le Q^f/\lambda \\ \gamma_i^f\left[p_i - c - \left(z_{it}^f + b_i\right)\left(T^f - t\right)\right] - \left(1-\gamma_i^f\right)a_i & t > Q^f/\lambda \end{cases} \quad (3.12\text{-a})$$

and the cycle profit is similar to (3.3)

$$\Pi^f\left(Z^f, Q^f \middle| T^f\right) = \sum_{i=1}^{N} \lambda_i \int_0^{T^f} \pi_i^f\left(\tau, z_{i\tau}^f, Q^f \middle| T^f\right) d\tau . \quad (3.12\text{-b})$$

The FCFS problem for the infinite horizon model is defined as

$$\frac{\Pi\left(Z^{f*}, Q^{f*} \middle| T^{f*}\right) - K}{T^{f*}} = \max_{T^f > 0}\left[\frac{\Pi\left(Z^{f*}, Q^{f*} \middle| T^f\right) - K}{T^f}\right] \quad (3.13\text{-a})$$

where

$$\Pi\left(Z^{f*}, Q^{f*} \middle| T^f\right) = \max_{Z^f, Q^f}\left[\Pi\left(Z^f, Q^f \middle| T^f\right)\right] \quad (3.13\text{-b})$$

$$\text{s. t.} \quad \left(1 - \alpha_i/\beta_i\right) \ge z_i^f \ge 0 \quad \text{and} \quad Q^f \ge 0$$

Corollary of Proposition 3.1. *Given* $\left(Q^f, T^f\right)$, *the optimal price discount and waiting rate for the FCFS policy are uniquely determined as*

$$z_{it}^{f*} = \max\left[\min\left(\frac{p_i + a_i - c}{2\left(T^f - t\right)} - \frac{b_i}{2} - \frac{\alpha_i}{2\beta_i}, \frac{1-\alpha_i}{\beta_i}\right), 0\right] \text{ and } \gamma_{it}^{f*} = \alpha_i + \beta_i z_{it}^{f*} \quad (3.14)$$

■

Let $w_i^f\left(t, z_{it}^{f*} \middle| T^f\right)$ denote be the prompt service welfare for class i demand. We have

$$w_i^f\left(t, z_{it}^f \middle| T^f\right) = \left(1 - \gamma_{it}^f\right)\left(p_i + a_i - c\right) + \gamma_i\left(z_{it}^f + b_i\right)\left(T^f - t\right) - ht$$

Let t^{f*} denote the time at which the average prompt service welfare given z_{it}^{f*} is zero, i.e. $\sum_{i=1}^{N}\lambda_i w_i^f\left(t^{f*}, z_{it_i}^{*}, Q^f \middle| T\right)=0$, called the *allocation threshold time* of the FCFS policy. Or, $t^{f*}=T$ if $\sum_{i=1}^{N}\lambda_i w_i^f\left(t^f, z_{it_i}^{*}, Q^f \middle| T\right)\geq 0$ for all $t^f\in[0,T]$.

Corollary of Proposition 3.2. *Given T^f and applied Z^{f*}, the prompt service welfare decreases in t^f. The allocation threshold time is uniquely determined as*

$$t^{f*}=\min\left[\frac{\sum_{i=1}^{N}\lambda_i\left(1-\gamma_{it^{f*}}^{f*}\right)\left(p_i+a_i-c\right)+T^f\sum_{i=1}^{N}\lambda_i\gamma_{it^{f*}}^{f*}\left(z_{it^{f*}}^{f*}+b_i\right)}{h+\sum_{i=1}^{N}\lambda_i\gamma_{it^{f*}}^{f*}\left(z_{it^{f*}}^{f*}+b_i\right)},T\right] \qquad (3.15\text{-a})$$

The optimal base stock level is uniquely determined as

$$Q^{f*}=\lambda t^{f*} \qquad (3.15\text{-b})$$

The optimal per-time-unit backorder cost at reorder point is

$$R^{f*}=\sum_{i=1}^{N}\lambda_i^*\int_{t^{f*}}^{T^{f*}}\gamma_{i\tau}^{f*}\left(z_{i\tau}^{f*}+b_i\right)d\tau \text{ and } r^{f*}=\sum_{i=1}^{N}\lambda_i^*\int_{t^{f*}}^{T^{f*}}\gamma_{i\tau}^{f*}d\tau. \qquad (3.15\text{-c})$$

■

Corollary of Proposition 3.3. *Applying the optimal policy $\left(Z^{f*},Q^{f*}\middle|T^f\right)$, the optimal cycle interval T^{f*} is uniquely determined by*

$$T^f\left(\partial\Pi^f/\partial T^f\right)-\Pi^f+K=0 \qquad (3.16\text{-a})$$

where

$$\begin{aligned}&T^f\left(\partial\Pi^f/\partial T^f\right)-\Pi^f+K\\&=T^f\left(\sum_{i=1}^{N}\lambda_i\left(p_i-c\right)-hQ^{f*}\right)-\sum_{i=1}^{N}\lambda_i\int_0^{T^f}\pi_i^f\left(\tau,z_{i\tau}^{f*},Q^{f*}\middle|T^f\right)d\tau+K\end{aligned} \qquad (3.16\text{-b})$$

decreases in T^f.

■

3.4.2. Comparison of the RD Solution to the FCFS Solution

Proposition 3.4. *Given (Q,T), the inventory rationing increases the average profit rate and the increment is*

$$\Delta_{\pm A}\Pi = (1/T)\sum_{i=1}^{N}\lambda_i \int_{t^{*f}}^{t_i^*} w_i^f\left(\tau, z_{i\tau}^{*f}, Q^f \middle| T\right) d\tau \tag{3.17}$$

Proof. Compare the conditional optimal policies $\left(t^{f*}, Z^{f*} \middle| Q, T\right)$ and $\left(Y^*, Z^* \middle| Q, T\right)$. The RD policy increases the cycle profit, since it allocates the units to the demands with the highest prompt service welfares. The RD policy only change the allocation of class i customers arriving between t^{f*} and t_i^*, that results (3.17). ■

The inventory allocation shifts some units from some customer classes with lower prices to those with higher prices. This kind of shifting is determined by prompt service benefit, not by the number of units. The base stock level may increase if the incremental number of prompt service for higher profit customers is higher than the prompt service reduction among lower profit customers. Or, the base stock level may decrease in the otherwise situation. Moreover, the cycle time may also be increased or decreased, though the situation is more complex.

3.4.3 Role of Price Discounting

The problem of not using price discount is equivalent to the problem with zero discount sensitive degree. We present the solution in the following proposition.

Proposition 3.5. *For the RD problem without price discounts, i.e.* $z_{it} = 0$ *(or* $\beta_i = 0$*), the RD policy is simplified as*

$$t_i^* = \min\left[\frac{(1-\alpha_i)(p_i + a_i - c) + \alpha_i b_i T^*}{h + \alpha_i b_i}, T^*\right] \tag{3.18-a}$$

$$Q^* = \sum_{i=1}^{N} \lambda_i t_i^* \tag{3.18-b}$$

$$R^* = \sum_{i=1}^{N} \lambda_i \alpha_i b_i \left(T^* - t_i^*\right) \text{ and } r^* = \sum_{i=1}^{N} \lambda_i \alpha_i \left(T^* - t_i^*\right) \tag{3.18-c}$$

where T^* *satisfies*

$$\sum_{i=1}^{N}\lambda_i\left[\begin{array}{l}(1-\alpha_i)(p_i+a_i-c)(T-t_i^*)\\ -(h+b_i\alpha_i)Tt_i^*+\frac{b_i\alpha_i}{2}T^2+\frac{h+b_i\alpha_i}{2}(t_i^*)^2\end{array}\right]+K=0 \qquad (3.18\text{-}d)$$

For the completely backorder case, i.e. $\alpha_i=1$*, the cycle time is*

$$T^*=\sqrt{2K\Big/\sum_{i=1}^{N}\frac{\lambda_i h b_i}{(h+b_i)}} \qquad (3.18\text{-}e)$$

Proof. Given a cycle time T, we have (3.18-a), (3.18-b), (3.18-c) by simplifying (3.6), (3.7), (3.8) directly. The cycle profit is equal to the selling revenue minus the holding, backorder and lost sales costs as following

$$\Pi=\sum_{i=1}^{N}\lambda_i\left[\begin{array}{l}(p_i-c)t^*+\alpha_i(p_i-c)(T-t^*)-\frac{h}{2}(t_i^*)^2\\ -\frac{b_i\alpha_i}{2}(T-t_i^*)^2-(1-\alpha_i)a_i(T-t_i^*)\end{array}\right]$$

Some algebra similar to (3.9) and (3.10) results (3.18-d). Given $\alpha_i=1$, we have $t_i^*=\dfrac{\alpha_i b_i T^*}{h+\alpha_i b_i}$ and (3.18-e) from (3.18-a) and (3.18-d). ■

Proposition 3.6. *For the RD problem without price discounts, i.e.* $z_{it}=0$ *(or* $\beta_i=0$*), given a cycle time* $T>\max_{1\le i\le N}\{(1-\alpha_i)(p_i+a_i-c)/h\}$*, there exists a balance among the accumulative holding cost at the beginning of cycle, the accumulate backorder cost at reorder point* R^* *and the average inventory cost.*

$$hQ^*=R^*+\sum_{i=1}^{N}\lambda_i(p_i+a_i-c)(1-\alpha_i)=\left(\sum_{i=1}^{N}\lambda_i(p_i-c)-\Pi^*/T\right)+K/T \qquad (3.19\text{-}a)$$

For completely backorder case, i.e. $\alpha_i=1$ *, the equation is simplified as*

$$hQ^*=R^*=\left(\sum_{i=1}^{N}\lambda_i(p_i-c)-\Pi^*/T\right)+K/T. \qquad (3.19\text{-}b)$$

Proof: Adding an incremental time to an cycle time, the incremental profit is the profit at $(t_i^*,t_i^*+\Delta T]$ with $w_i\left(t_i^*,z_{it_i^*}^*\middle|T\right)=0$. We have from (3.2) and $\gamma_{it}=\alpha_i$

$$\Delta\Pi/\Delta T = \sum_{i=1}^{N}\lambda_i\alpha_i\left(p_i - c\right) - \sum_{i=1}^{N}\lambda_i a_i(1-\alpha_i) - \sum_{i=1}^{N}\lambda_i\alpha_i b_i\left(T - t_i^*\right)$$
$$= \sum_{i=1}^{N}\lambda_i\left(p_i - c\right) - h\sum_{i=1}^{N}\lambda_i t_i^*$$

Plugging in (3.7), (3.8) and (3.9), we obtain

$$\left(\Pi^* - K\right)/T = \sum_{i=1}^{N}\lambda_i\alpha_i\left(p_i - c\right) - \sum_{i=1}^{N}\lambda_i a_i(1-\alpha_i) - R^* = \sum_{i=1}^{N}\lambda_i\left(p_i - c\right) - hQ^*$$

and results (3.19-a) and follows (3.19-b) if $\alpha_i = 1$. ■

In (3.19-a), hQ^* is the per-time-unit holding cost at the beginning of cycle, $\sum_{i=1}^{N}\lambda_i\left(p_i + a_i - c\right)(1-\alpha_i)$ is the per-time-unit for lost sales, $\sum_{i=1}^{N}\lambda_i\left(p_i - c\right) - \Pi^*/T$ is the average inventory cost and K/T is the average setup cost.

For the general RD problem, some results in (3.19-a) still hold by the analysis on the marginal cycle time, such as

$$hQ^* = \sum_{i=1}^{N}\lambda_i\left(p_i - c\right) - \Pi/T + K/T. \tag{3.20-a}$$

However, since the optimal price discount z_{it}^* increases with the arriving time from Proposition 1, the result for the total per-unit-time backorder cost only hold in a weaker way from (3.8) and the monotony of z_{it}^* and γ_{it}^*

$$\sum_{i=1}^{N}\lambda_i\gamma_{iT}^*\left(z_{iT}^* + b_i\right)\left(T - t_i^*\right) \geq R^* \geq \sum_{i=1}^{N}\lambda_i\gamma_{it_i^*}^*\left(z_{it_i^*}^* + b_i\right)\left(T - t_i^*\right). \tag{3.20-b}$$

Since the prompt service benefit at t_i^* is zero, we get from (3.20-b) and (3.10)

$$R^* \geq \sum_{i=1}^{N}\lambda_i\left[ht_i^* - \left(1 - \gamma_{it_i^*}^*\right)\left(p_i + a_i - c\right)\right] = hQ^* - \sum_{i=1}^{N}\lambda_i\left(1 - \gamma_{it_i^*}^*\right)\left(p_i + a_i - c\right) \tag{3.20-c}$$

Intuitively, when the demand is closer to the end of cycle, the firm offers the customers higher discounts to increase the waiting rates. Thus, the accumulative backlogging cost at the end of cycle is increased.

The price discounts reduce the lost sales penalty and increase the cycle profit. According to (3.9), the marginal profit increment and the cycle profit increment have the opposite effects on the cycle time increment. Thus, the cycle time may increase as well as decrease. So is the associated base stock level.

3.4.4. Static Analysis

We report the static analysis for the allocation threshold time and cycle time in the following propositions.

Proposition 3.7. *Given a cycle time T, the static analysis for the allocation threshold time: a) t_i^* increases in p_i; b) t_i^* increases in a_i; c) t_i^* increases in b_i; d) t_i^* decreases in h; e) t_i^* decreases in β_i; f) t_i^* increases in α_i, if*

$$\left\{0 < z_{it_i^*}^* < \min\left(b_i, \frac{1-\alpha_i}{\beta_i}\right)\right\} \cup \left\{z_{it_i^*}^* = 0 \text{ and } b_i\left(T - t_i^*\right) \geq \left(p_i + a_i - c\right)\right\};$$

Otherwise, t_i^* decreases in α_i.

Comments The results in a)-e) are neat and intuitive. In f), a higher initial waiting rate may increase the accumulative backlogging cost when the customer arriving around the allocation threshold time. Then, the firm may offer more prompt service instead of saving the cost for lost sales.

Proof. Observe

$$w_i\left(t, z_{it}^*\middle|T\right)$$

$$=\begin{cases} (1-\alpha_i)(p_i+a_i-c)+\alpha_i b_i(T-t)-ht & z_{it}^*=0 \\ \left[\begin{aligned} &\left(1-\beta_i\left(\frac{p_i+a_i-c}{2(T-t)}-\frac{b_i}{2}+\frac{\alpha_i}{2\beta_i}\right)\right)(p_i+a_i-c) \\ &+\beta_i\left(\frac{p_i+a_i-c}{2(T-t)}-\frac{b_i}{2}+\frac{\alpha_i}{2\beta_i}\right)\left(\frac{p_i+a_i-c}{2(T-t)}+\frac{b_i}{2}-\frac{\alpha_i}{2\beta_i}\right)(T-t)-ht \end{aligned}\right] & 0<z_{it}^*<\frac{1-\alpha_i}{\beta_i} \\ \left(\frac{1-\alpha_i}{\beta_i}+b_i\right)(T-t)-ht & z_{it}^*=\frac{1-\alpha_i}{\beta_i} \end{cases}$$

$$=\begin{cases} (1-\alpha_i)(p_i+a_i-c)+\alpha_i b_i(T-t)-ht & z_{it}^*=0 \\ \left[\begin{aligned} &-\beta_i\frac{(p_i+a_i-c)^2}{4(T-t)}+\left(1-\frac{\alpha_i}{2}+\frac{\beta_i b_i}{2}\right)(p_i+a_i-c) \\ &-\beta_i\left(\frac{b_i}{2}-\frac{\alpha_i}{2\beta_i}\right)^2(T-t)-ht \end{aligned}\right] & 0<z_{it}^*<\frac{1-\alpha_i}{\beta_i} \\ \left(\frac{1-\alpha_i}{\beta_i}+b_i\right)(T-t)-ht & z_{it}^*=\frac{1-\alpha_i}{\beta_i} \end{cases}$$

Consider the first order condition on p_i, we have

$$\frac{\partial w_i\left(t, z_{it}^*\middle|T\right)}{\partial p_i}=\left(\begin{cases} (1-\alpha_i) & z_{it}^*=0 \\ -\beta_i\frac{p_i+a_i-c}{2(T-t)}+1-\frac{\alpha_i}{2}+\frac{\beta_i b_i}{2} & 0<z_{it}^*<\frac{1-\alpha_i}{\beta_i} \\ 0 & z_{it}^*=\frac{1-\alpha_i}{\beta_i} \end{cases}\right)=1-\gamma_i\geq 0$$

The result in a) holds, since the value of $w_i\left(t, z_{it}^*\middle|T\right)$ as well as the solution of $w_i\left(t, z_{it}^*\middle|T\right)=0$ increases in p_i. Similarly, the result in b) holds from

$$\frac{\partial w_i\left(t, z_{it}^*\middle|T\right)}{\partial a_i}=\frac{\partial w_i\left(t, z_{it}^*\middle|T\right)}{\partial p_i}\geq 0.$$

The result in c) holds from

$$\frac{\partial w_i\left(t, z_{it}^*\middle|T\right)}{\partial b_i} = \begin{cases} \alpha_i\left(T-t\right) & z_{it}^* = 0 \\ \left[\left(\frac{\beta_i}{2}\left(p_i + a_i - c\right)\right) - \beta_i\left(\frac{b_i}{2} - \frac{\alpha_i}{2\beta_i}\right)\left(T-t\right)\right] & 0 < z_{it}^* < \frac{1-\alpha_i}{\beta_i} \\ \left(T-t\right) & z_{it}^* = \frac{1-\alpha_i}{\beta_i} \end{cases}$$

$$= \begin{cases} \alpha_i\left(T-t\right) & z_{it}^* = 0 \\ \beta_i\left(z_{it}^* + \frac{\alpha_i}{\beta_i}\right)\left(T-t\right) & 0 < z_{it}^* < \frac{1-\alpha_i}{\beta_i} \\ \left(T-t\right) & z_{it}^* = \frac{1-\alpha_i}{\beta_i} \end{cases}$$

$$\geq 0$$

The result in d) holds from

$$\frac{\partial w_i\left(t, z_{it}^*\middle|T\right)}{\partial h} = -t \leq 0 .$$

The result in e) holds from

$$\frac{\partial w_i\left(t_i^*, z_{it_i^*}^* \middle| T\right)}{\partial \beta_i}$$

$$= \begin{cases} 0 & z_{it_i^*}^* = 0 \\ \left[\begin{array}{l} -\frac{(p_i + a_i - c)^2}{4(T-t)} + \frac{b_i}{2}(p_i + a_i - c) \\ -\left(\frac{b_i}{2} - \frac{\alpha_i}{2\beta_i}\right)^2 (T-t) - \frac{\alpha_i}{\beta_i}\left(\frac{b_i}{2} - \frac{\alpha_i}{2\beta_i}\right)(T-t) \end{array}\right] & 0 < z_{it_i^*}^* < \frac{1-\alpha_i}{\beta_i} \\ -\left(\frac{1-\alpha_i}{\beta_i^2}\right)(T-t) & z_{it_i^*}^* = \frac{1-\alpha_i}{\beta_i} \end{cases}$$

$$= \begin{cases} 0 & z_{it}^* = 0 \\ \left[-\frac{(p_i + a_i - c)^2}{4(T-t)} + \frac{b_i}{2}(p_i + a_i - c) + \left[\left(\frac{\alpha_i}{2\beta_i}\right)^2 - \left(\frac{b_i}{2}\right)^2\right](T-t)\right] & 0 < z_{it}^* < \frac{1-\alpha_i}{\beta_i} \\ -\left(\frac{1-\alpha_i}{\beta_i^2}\right)(T-t) & z_{it}^* = \frac{1-\alpha_i}{\beta_i} \end{cases}$$

$$= \begin{cases} 0 & z_{it}^* = 0 \\ \left[-\left(z_{it}^* + \frac{\alpha_i}{2\beta_i}\right)^2 + \left(\frac{\alpha_i}{2\beta_i}\right)^2\right](T-t) & 0 < z_{it}^* < \frac{1-\alpha_i}{\beta_i} \\ -\left(\frac{1-\alpha_i}{\beta_i^2}\right)(T-t) & z_{it}^* = \frac{1-\alpha_i}{\beta_i} \end{cases}$$

$$\leq 0$$

Focus on the effect at time t_i^*, the result in f) holds from

$$\frac{\partial w_i\left(t, z_{it}^*\middle|T\right)}{\partial \alpha_i} = \begin{cases} -\left(p_i + a_i - c\right) + b_i\left(T - t\right) & z_{it}^* = 0 \\ -\frac{1}{2}\left(p_i + a_i - c\right) - \left(\frac{b_i}{2} - \frac{\alpha_i}{2\beta_i}\right)\left(T - t\right) & 0 < z_{it}^* < \frac{1-\alpha_i}{\beta_i} \\ -\frac{1}{2\beta_i}\left(T - t\right) & z_{it}^* = \frac{1-\alpha_i}{\beta_i} \end{cases}$$

$$= \begin{cases} -\left(p_i + a_i - c\right) + b_i\left(T - t\right) & z_{it}^* = 0 \\ \left(-z_{it}^* + b_i\right)\left(T - t\right) & 0 < z_{it}^* < \frac{1-\alpha_i}{\beta_i} \\ -\frac{1}{2\beta_i}\left(T - t\right) & z_{it}^* = \frac{1-\alpha_i}{\beta_i} \end{cases}$$

■

The static analysis for base stock follows directly from Proposition 3.7 and $Q^* = \sum_{i=1}^{N} \lambda_i t_i^*$. However, the static analysis for cycle time is very complex as illustrated in the following two notes.

Note 3.1. *The static analysis for the cycle time on the price* p_i.

Analysis: Observe the first order condition on $T\left(\partial\Pi/\partial T\right) - \Pi$, we have similar to the prove in Proposition 3.7

$$\frac{\Delta\left[T\left(\partial\Pi/\partial T\right) - \Pi + K\right]}{\Delta p_i}$$

$$= \sum_{i=1}^{N} \lambda_i \left[T\left(\frac{\Delta\left(p_i - c - ht_i^*\right)}{\Delta p_i}\right) - \int_0^T \frac{\Delta\pi_i\left(\tau, y_{i\tau}^*, z_{i\tau}^*\middle|T\right)}{\Delta h} d\tau\right]$$

$$= \sum_{i=1}^{N} \lambda_i \left[T\left(1 - h\frac{\Delta t_i^*}{\Delta p_i}\right) - \int_0^{t_i^*} d\tau - \int_{t_i^*}^{T} \gamma_{i\tau} d\tau\right]$$

$$= \sum_{i=1}^{N} \lambda_i \left[\int_{t_i^*}^{T} \left(1 - \gamma_{i\tau}\right) d\tau - Th\frac{\Delta t_i^*}{\Delta p_i}\right]$$

Since t_i^* increases in p_i from Proposition 3.7 a), we have $\frac{\Delta t_i^*}{\Delta p_i} \geq 0$. However, the value of $\frac{\Delta t_i^*}{\Delta p_i}$ is not clear, since t_i^* is determined by a complex implied function. Thus, we do not know whether or not $\frac{\Delta\left[T(\partial\Pi/\partial T)-\Pi+K\right]}{\Delta p_i}$ is positive and whether or not T increases in p_i.

■

Note 3.2. The static analysis for the cycle time on the price h.

Analysis: We have similar to Remark 3.2

$$\frac{\Delta\left[T(\partial\Pi/\partial T)-\Pi+K\right]}{\Delta h}$$

$$=\sum_{i=1}^{N}\lambda_i\left[T\left(\frac{\Delta\left(p_i-c-ht_i^*\right)}{\Delta h}\right)-\int_0^T\frac{\Delta\pi_i\left(\tau,y_{i\tau}^*,z_{i\tau}^*\middle|T\right)}{\Delta h}d\tau\right]$$

$$=\sum_{i=1}^{N}\lambda_i\left[T\left(-t_i^*-h\frac{\Delta t_i^*}{\Delta h}\right)-\int_0^{t_i^*}(-\tau)d\tau\right]$$

$$=\sum_{i=1}^{N}\lambda_i\left[-\left(T-\frac{t_i^*}{2}\right)t_i^*-Th\frac{\Delta t_i^*}{\Delta h}\right]$$

Since t_i^* decreases in h from Proposition 3.7 d), Thus, similar to the reason in Remark 3.1, we do not know whether or not T decreases in h.

■

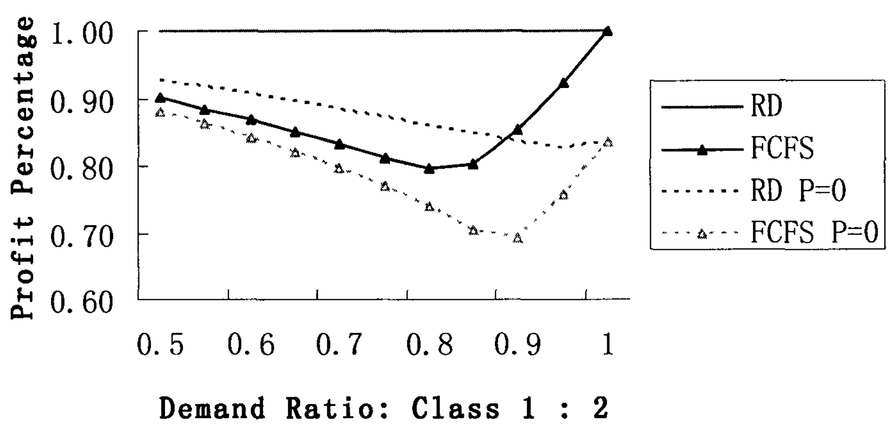
3.4.5. Managerial Insights and Numerical Analysis

We experiment with the following data set: $c=1$, $h=0.5$, $K=200$, $N=2$, $\lambda_1+\lambda_2=10$, $a=(0.5,5)$, $b=(0.01,0.1)$, $\alpha=(0.8,0.0)$, $\beta=(10,0.1)$ and $p=(10,20)$. Increasing the percentage of class 1 customer from 50% to 100%, we report the profits as the percentage of Rd policy, the fill rates of all demands, the base stock level and the cycle time in Figures 3.1 (a)__(d).

Figure 3.1 (a) illustrates both rationing and price discounting increases the profit significantly. The RD policy can increase the profit up to 30% from the FCFS policy without price discounting. In Figure 3.1 (b), the price discount improves the total fill rate from 80% to 100%.

As shown in Figure 3.1 (c) and 3.1 (d), the RD policy reduces the base stock levels in this example, but the cycle time may be increased or decreased at different demand ratio by the effect of rationing and pricing discount. For the policies without price discounts, the cycle times may be shorten for the sake of reducing the lost sales penalty or holding cost when there are plenty of high profit customers. When the ratio of high profit customer becomes very small, the firm may focus on the service for the lower profit customers. For the policy without price discount, the accumulative backlogging cost is low and the firm may increase the cycle time to reduce the setup cost. The effect of rationing on the cycle time is at similar situation.

Figure 3.1. (d) also tell us that the cycle time may be sensitive to the demand ratio. So, it may not easy to estimate an accurate cycle time if ignoring this kind of sensitivity. The optimal cycle time may be far away from the related solution of single class problem. This implies, the performance of multi-class problem is quite different from the traditional single class EOQ model.



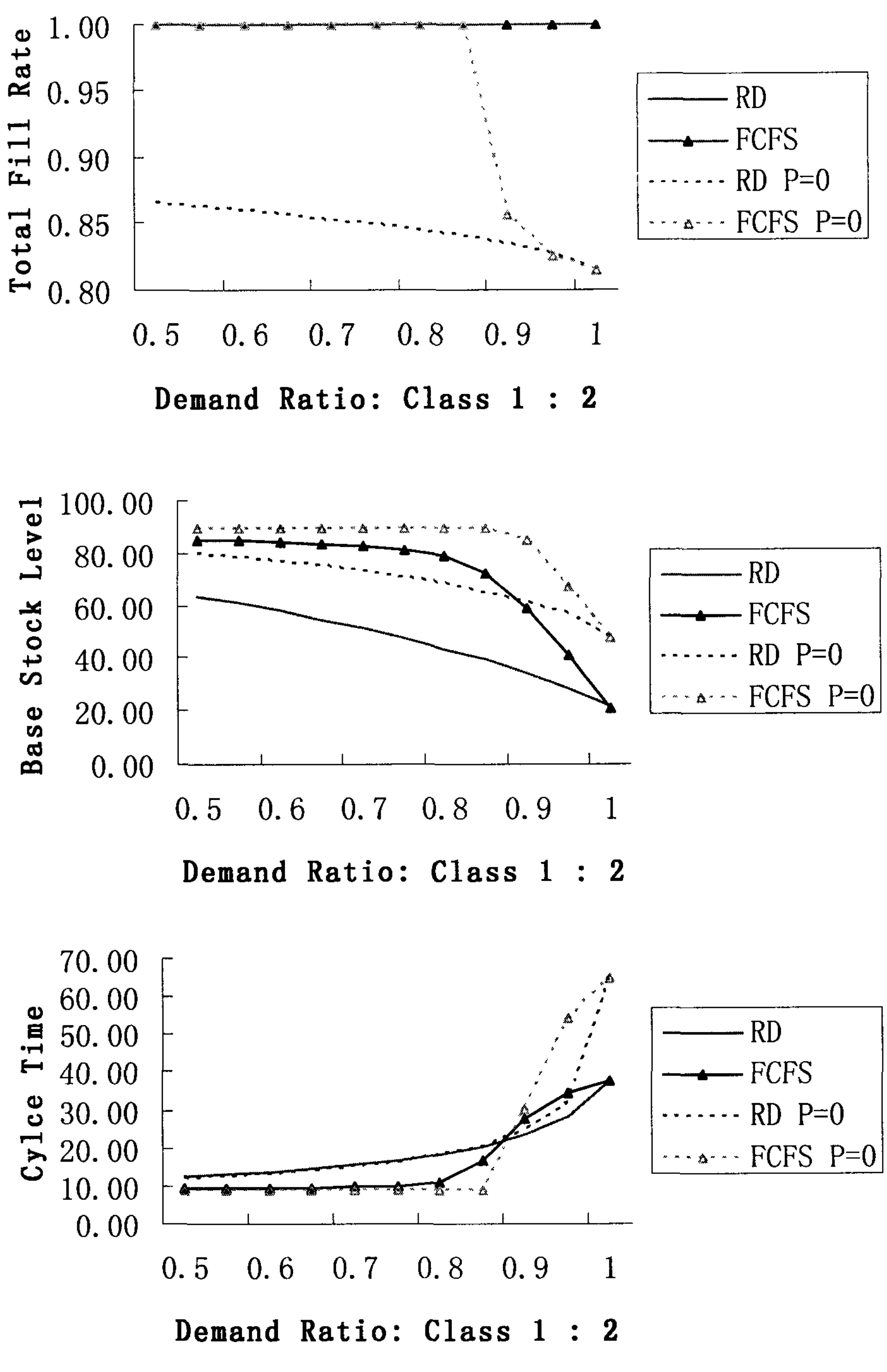


Figure 3.1 Vary the percentage of class 1 customer from 50% to 100% and compare: (a). Perfect as Percentage; (b) Total Fill Rate; (c). Base Stock Level; (d). Cycle Time.

3.5 Conclusions

The model in this paper considers the partial backlogging case for multiple customer class inventory system where the likelihood of backlogging is linearly dependent on a discount offered. We focus on the deterministic demand problems in a generalized EOQ model.

Previous research (Moon and Kang 1998) provides an optimal rationing level in a continuous review policy for an EOQ model with two customer classes and the unfulfilled demands completely backordered, while the value of order quantity and reorder point are given constants.

In this paper, we propose an optimal rationing and discounting policy for the EOQ model with multi-class customers. We use the intuitive concept of prompt service welfare to find the optimal price discounts and the optimal rationing policy given the cycle time. Then, the optimal order policy is determined from the first order condition on the cycle time. The numerical analysis illustrates that both inventory rationing and price discounting can increase the average profit and the customer fill rates significantly by comparing the results from the no price discounting policy and the naïve first come / first serve policy. In the future research, we can apply the generalized EOQ policies for the time varying demand and multi-stage problems. It is also interesting and hopeful to apply this kind of approach to solve the corresponded problems with stochastic demands in the future research.

3.6 References

K. D. Cattani and G. C. Souza, 2002. Inventory Rationing and Shipment Flexibility Alternatives for Direct Market Firms. *Production and Operations Management*. 11, 4, 441-457.

F. Chen, 2001. Market Segmentation, Advanced Demand Information, and Supply Chain Performance. *Manufacturing & Service Operations Management*. 3, 1, 53-67.

K. L. Cheung, 1998. A Continuous Review Inventory Model with a Time Discount. *IIE Transactions*. 30, 747-757.

M. A. Cohen, P. R. Kleindorfer and H. L. Lee, 1988. Service Constrained (s,S) Inventory Systems with Priority Demand Classes and Lost Sales. *Management Science*. 34, 482-499.

G. A. DeCroix and A. Arreola-Risa, 1998. On Offering Economic Incentives to Backorder. *IIE Transactions*. 30, 715-721.

R. Dekker, M. J. Kleijn and P. J. de Rooij, 1998. A spare parts stocking system based on equipment criticality. *International Journal of Production Economics*. 56, 69-77.

G. V. Deshpande, M. A. Cohen and K. Donohue, 2003. A Threshold Inventory Rationing Policy for Service-Differentiated Demand Classes. *Management Science*. 49, 6, 683-703.

Q. Ding, P. Kouvelis and J. Milner, 2003. Dynamic Pricing through Customer Discounts for Optimizing Multi-Class Customers Demand Fulfillment. *Working Paper*. Olin School of Business, Washington University.

K. C. Frank, R. Q. Zhang and I. Duenyas, 2003. Optimal Policies for Inventory Systems with Priority Demand Classes. *Operations Research*. 51, 6, 993-1002.

I. L. Gale and T. J. Holmes, 1993. Advance-Purchase Discounts and Monopoly Allocation of Capacity. *American Economic Review*. 83, 1,135-146.

M. J. Kleijn and R. Dekker, 1998. An Overview of Inventory Systems with Several Demand Classes. *Report EI9838/A*, Econometric Institute, Erasmus University Rotterdam, the Netherlands.

P. Melchiors, R. Dekker and M. Kleijn, 1998. Inventory Rationing in An (s, Q) Inventory Model with Lost Sales and Two Demand Classes. *Report EI9837/A*, Econometric Institute, Erasmus University Rotterdam, the Netherlands.

I. Moon and S. Kang, 1998. Rationing Policies for Some Inventory System. *the Journal of the Operational Research Society*, 49 (5), 509-518.

D. M. Topkis, 1968. Optimal Ordering and Rationing Policies in A Nonstationary Dynamic Inventory Model with n Demand Classes. *Management Science.* 15, 160-176.

Y. Wang, M. A. Cohen and Y.-S. Zheng, 2002. Differentiating Customer Service on the Basis of Delivery Lead Times. *IIE Transactions*. 34, 11, 979-989.

P. H. Zipkin, 2000. Foundations of Inventory Management. McGraw-Hill, New York.